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PREFACE

The book is intended to meet the requirements of students at Indian Universities. The subject matter has been presented in a manner understandable to the average student and every attempt has been made to make it clear and lucid. There are a number of solved examples, easy as well as difficult, illustrating the technique of solution.

The order of the chapters aims at giving a rational classification of the topics dealt with. A separate chapter has been devoted to the General Equation of the Second Degree, wherein loci common to special conics have been obtained.

The examples have been selected mostly from standard works on the subject and from Examination Papers of Indian as well as British Universities. Questions set at the various Public Service Examinations have been included in the text as far as possible.

I wish to acknowledge my indebtedness to the authors of various treatises on Co-ordinate Geometry many of which have been freely consulted while preparing the present volume.

My thanks are due to the authorities of London University for their permission to include questions from the Examination Papers of their University. I am grateful also to my learned teacher Prof. A. N. Singh, D. Sc. for his suggestions and advice, and to my colleagues in the Department of Mathematics, Lucknow University, for their help in the preparation of the book.

Any suggestions for improvement will be gratefully acknowledged.

LUCKNOW,
MARCH, 1949.

RAM BALLABH.

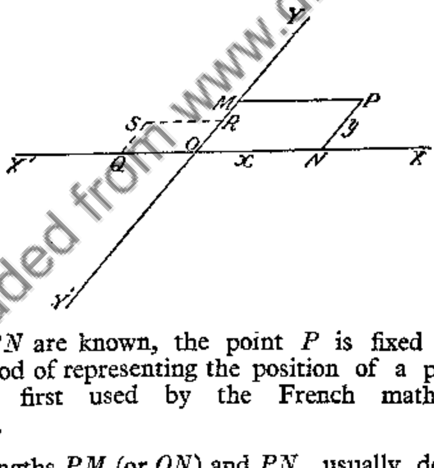
CONTENTS

CO-ORDINATE GEOMETRY

CHAPTER I

CO-ORDINATE SYSTEMS

1·1. Cartesian Co-ordinates. Let $X'OX$, $Y'OY$ be two fixed straight lines in a plane, intersecting at O. Let P be any point on the plane. Let PM, PN be drawn parallel to $X'OX$, $Y'OY$ respectively. If the lengths



PM and PN are known, the point P is fixed **uniquely.** This method of representing the position of a point in a plane was first used by the French mathematician, des Cartes.

The lengths PM (or ON) and PN, usually denoted by x, y respectively which determine the position of the point P with reference to the lines $X'OX$, $Y'OY$ are called the **Cartesian co-ordinates,** or briefly, the **co-ordinates** of P. The lines $X'OX$, $Y'OY$ are called the **co-ordinate axes,** $X'OX$ being the axis of x, $Y'OY$ the axis of y. The point of intersection O of the axes is called the **origin.** ON

(or x) is called the **abscissa,** and PN (or y) the **ordinate** of the point P which is then called the point (x, y). The abscissa and the ordinate are also called the x and y co-ordinates respectively. The x co-ordinate is always written first.

The axes are said to be **rectangular or oblique** according as the angle between them is a right angle or different from a right angle.

1·2. Sign convention. The conventions regarding signs are the same as those used in Trigonometry. The plane is supposed to be divided into four quadrants by the axes. The quadrant containing the angle XOY is called the **first quadrant,** the one containing the angle YOX' the **second quadrant** and the ones containing the angles $X'OY'$ and $Y'OX$ the **third** and the **fourth** quadrants respectively. The following table gives the signs of x and y, the co-ordinates of a point in several quadrants.

	x co-ordinate	y co-ordinate
First Quadrant	+	+
Second Quadrant	−	+
Third Quadrant	−	−
Fourth Quadrant	+	−

Thus we see that OX and OY are the positive directions of the x and y axes and OX', OY' their negative directions.

To plot the point $(-2, 1)$ we measure a length OQ 2 units along OX' and OR 1 unit along OY. The point S which is the point of intersection of the sides QS, RS of the parallelogram $OQSR$ is the required point.

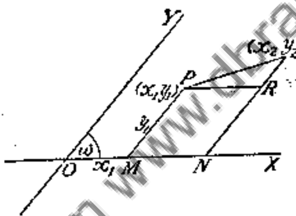
Examples 1. Plot the following points on a graph paper and see that they lie on a straight line

$$(4, 6), (-2, 0), (-6, -4), (-1, 1).$$

2. Plot the points (1, 2) (3, 8) (17, −6) and see that they are equidistant from the point (11, 2). Show that the radius of the circle through the first three points is 10 units.

1·3. Distance between two given points.

Let P, Q be the two given points with co-ordinates (x_1, y_1), (x_2, y_2) respectively and let the angle XOY be ω. Draw PM, QN parallel to the y—axis and PR parallel to



the x—axis. Then $PR = MN = x_2 - x_1$ and $QR = NQ - NR = NQ - PM = y_2 - y_1$. Also the angle PRQ is equal to $180° - \omega$. Hence from the triangle PQR,

$$PQ^2 = PR^2 + QR^2 - 2PR.QR \cos P\hat{R}Q$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos\omega.$$

In the case of rectangular axes, $\omega = 90°$ making $\cos\omega$ zero and PQ^2 is then equal to

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

The distance PQ between the given points is thus

$$\pm \sqrt{(\mathbf{x_2} - \mathbf{x_1})^2 + (\mathbf{y_2} - \mathbf{y_1})^2}$$

for rectangular axes.

It is customary to express the distance with the plus sign. In the case of straight lines drawn parallel to co-ordinate axes the length is taken to be positive or negative according as it is measured in the positive or negative sense of the axes. While measuring distances between points in the same straight line, the same direction should be considered positive throughout.

Corollary. The distance of the point (x, y) from the origin is $\sqrt{x^2+y^2+2xy\cos\omega}$ which for rectangular axes reduces to $\sqrt{x^2+y^2}$.

In the following examples the axes are **rectangular** :—

Examples 1. Find the centre of circle circumscribing the triangle whose vertices are the points (3, 4), (4, 2) and (5, 5).

The circum-centre is equidistant from each vertex of the triangle. If, therefore, (α, β) are the required co-ordinates, then

$$(\alpha-3)^2+(\beta-4)^2=(\alpha-4)^2+(\beta-2)^2$$
$$=(\alpha-5)^2+(\beta-5)^2,$$

from which we have

$$2\alpha-4\beta+5=0,$$

and

$$\alpha+3\beta-15=0.$$

Solving, $\alpha=\frac{9}{2}, \beta=\frac{7}{2}.$

Hence the centre of the circum-circle is the point $(\frac{9}{2}, \frac{7}{2})$.

2. Show that the points (1, 0), (−2, 3) and (−3, −4) are the vertices of a right-angled triangle.

3. Show that the four points (3, 1), (4, 2), (3, 3) and (2, 2) are the angular points of a square.

4. Find the distances between the points $(a, a\sqrt{3})$, $(a, 3a\sqrt{3})$ and $(4a, 2a\sqrt{3})$ taken two at a time. Hence show that they are the vertices of an equilateral triangle of side $2a\sqrt{3}$.

Ans. $2a\sqrt{3}$

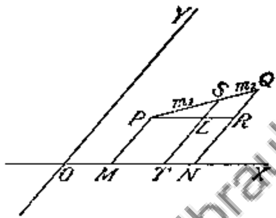
5. In any triangle ABC, prove that

$$AB^2+AC^2=2(AD^2+DC^2),$$

where D is the middle point of BC.

1·4. Co-ordinates of the point which divides in a given ratio the line joining two given points.

Let PQ be the straight line joining the two points P, Q whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively.



Let S be the point which divides PQ **internally** in the ratio $m_1 : m_2$, and let (x, y) be the co-ordinates of S.

Draw PM, ST, QN parallel to the y-axis and PLR parallel to the x-axis.

Then $PL = MT = x - x_1$, $LR = TN = x_2 - x$, and from the property of parallels, we get

$$PL : LR :: PS : SQ$$

i.e., $$\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2},$$

or $$x(m_2 + m_1) = m_1 x_2 + m_2 x_1$$

i.e., $$\mathbf{x} = \frac{\mathbf{m_1 x_2 + m_2 x_1}}{\mathbf{m_1 + m_2}}.$$

Similarly, $$\mathbf{y} = \frac{\mathbf{m_1 y_2 + m_2 y_1}}{\mathbf{m_1 + m_2}}.$$

If $m_1 = m_2$, *i.e.*, if S is the **point of bisection** of PQ, its co-ordinates are

$$\tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(y_1 + y_2).$$

If the point S divides PQ **externally** in the ratio $m_1 : m_2$, its co-ordinates will be seen to be

$$\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad \frac{m_1y_2 - m_2y_1}{m_1 - m_2}.$$

The student should not find any difficulty in proving the above result, for the distances PS and SQ are now measured in opposite senses. If m_1 is positive, m_2 becomes negative and *vice versa*. The result then follows at once from the co-ordinates of S for internal division. It should also be obvious that the point of external division lies to the right or left of P according as $m_1 >$ or $< m_2$.

Note. The above result and the result of § 1·3 have been proved for the case when the given points lie in the first quadrant, but they will be found to be true for points lying in any part of the plane provided the co-ordinates are taken with proper signs.

Examples 1. Find the co-ordinates of the points which divide the join of (4, 3) and (5, 7) internally and externally in the ratio 2 : 3.

Here $x_1 = 4$, $x_2 = 5$, $y_1 = 3$, $y_2 = 7$, and for internal division $m_1 = 2$, $m_2 = 3$.

Using the formula, the co-ordinates of the point of internal division are

$$\left(\frac{2.5 + 3.4}{2+3}, \frac{2.7 + 3.3}{2+3}\right), \text{ i.e., } \left(\tfrac{22}{5}, \tfrac{23}{5}\right).$$

For external division $m_1 = 2$, $m_2 = -3$, and the co-ordinates of the required point are

$$\left(\frac{2.5 - 3.4}{2-3}, \frac{2.7 - 3.3}{2-3}\right), \text{ i.e., } (2, -5).$$

2. Find the co-ordinates of the centroid of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3).

The centroid is the point of intersection of the medians of a triangle.

Ans. $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

3. G is the centroid of a triangle ABC and O any other point ; prove that

(i) $3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$

(ii) $OA^2+OB^2+OC^2=GA^2+GB^2+GC^2+3GO^2$.

Take AB and AC as the axes of x and y which will be inclined to each other at an angle A. The co-ordinates of the points A, B, C will be (o, o), (c, o) and (o, b) where c and b are respectively the lengths of AB and AC. The co-ordinates of G are $\left(\frac{1}{3}c, \frac{1}{3}b\right)$.

Using the formula of § 1·3 for oblique co-ordinates,

$$GA^2=\tfrac{1}{9}(b^2+c^2+2bc\cos A)$$

$$GB^2=\tfrac{1}{9}(b^2+4c^2-4bc\cos A)$$

and $$GC^2=\tfrac{1}{9}(4b^2+c^2-4bc\cos A).$$

Hence $$3(GA^2+GB^2+GC^2)=2(b^2+c^2-bc\cos A)$$
$$=b^2+c^2+(b^2+c^2-2bc\cos A)$$
$$=AC^2+AB^2+BC^2.$$

The student should find no difficulty in seeing that the particular choice of co-ordinate axes does not make the solution less general since the configuration is not affected by such choice.

The second part can be proved on similar lines.

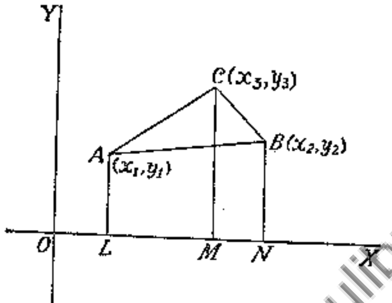
4. If a straight line AB is bisected at X, and also divided (i) internally, (ii) externally into two unequal segments at Y, show analytically that in either case

$$AY^2+YB^2=2(AX^2+XY^2).$$

5. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

Note. **The axes will henceforth be assumed to be rectangular unless the contrary is definitely expressed.**

1·5. The Area of a triangle in terms of the co-ordinates of its vertices.



Let A, B, C the vertices of the triangle have (x_1, y_1), (x_2, y_2) and (x_3, y_3) as their respective co-ordinates.

Draw the ordinates AL, BN, CM as in the figure.

Then,
$$\triangle ABC = \text{trapezium } AM + \text{trapezium } CN - \text{trapezium } AN$$
$$= \tfrac{1}{2}(AL+CM).LM + \tfrac{1}{2}(BN+CM).MN - \tfrac{1}{2}(AL+BN).LN$$
$$= \tfrac{1}{2}\{(y_1+y_3)(x_3-x_1)+(y_2+y_3)(x_2-x_3) - (y_1+y_2)(x_2-x_1)\}$$
$$= \tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

which is easily remembered in the determinantal form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area will be positive or negative according as it lies to the left or right in going round the triangle from (x_1, y_1) to (x_3, y_3) through the point (x_2, y_2). Thus the area of the triangle ABC is positive and of the same triangle written as ACB is negative. *Where numerical value of area is required, the sign may be disregarded.*

To the student familiar with vector notation the result should be capable of easy deduction as a determinant. The position vectors of the points A, B, C are respectively $x_1 \mathbf{i} + y_1 \mathbf{j}$, $x_2 \mathbf{i} + y_2 \mathbf{j}$, $x_3 \mathbf{i} + y_3 \mathbf{j}$, $\mathbf{i}$, $\mathbf{j}$ being the conventional unit vectors. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are then $(x_2 - x_1)\ \mathbf{i} + (y_2 - y_1)\ \mathbf{j}$ and $(x_3 - x_1)\ \mathbf{i} + (y_3 - y_1)\ \mathbf{j}$ respectively and their vector product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix},$$

which is equal to

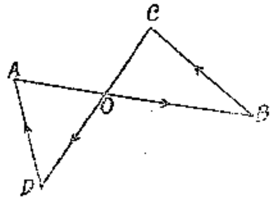
$$\mathbf{k}. \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \text{ or } \mathbf{k} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

k being the third conventional unit vector.

But the vector product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is also equal to $AB.\ AC \sin B\hat{A}C.\ \mathbf{k}$ or 2 (area of the triangle) $\mathbf{k}$. Hence the area of the triangle is equal to the determinant

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area of a polygon can be determined by breaking it up into triangles. The areas of the various triangles should be added up with proper signs. For example, in the



above figure the sense of description of the area $ABCD$ (sometimes called a quadrilateral) is shown by arrows. The area is easily seen to be the *difference* (not *sum*) of the areas of the triangles OBC and ODA.

1·6. Collinear Points. Three points lying on one straight line will clearly form a triangle of zero area. Hence the condition that the points (x_1, y_1), (x_2, y_2), (x_3, y_3) should be on a straight line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

We shall see in the Chapter on Straight Lines that this condition for collinearity of three points holds for oblique axes as well.

Examples 1. If (1, 1), (7, −3), (12, 2) and (7, 21) are the co-ordinates of the vertices of a quadrilateral, prove that its area is 132 square units. [*Roorkee, 1945*]

Denoting by A, B, C, D the angular points (1, 1), (7, −3), (12, 2) and (7, 21) respectively we can divide the quadrilateral into two triangles ABC and CDA such that the respective areas lie to the left of an observer going round the triangle

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 7 & -3 & 1 \\ 12 & 2 & 1 \end{vmatrix} = 25 \text{ sq. units}$$

$$\triangle CDA = \frac{1}{2}\begin{vmatrix} 12 & 2 & 1 \\ 7 & 21 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 107 \text{ sq. units}$$

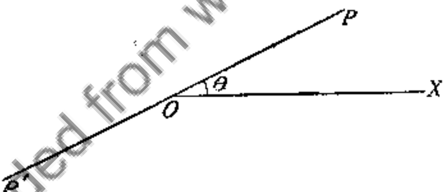
The area of the quadrilateral is therefore $(25+107) = 132$ sq. units.

2. Show that the area of the quadrilateral whose angular points taken in order are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is

$$\tfrac{1}{2}\{x_1(y_2-y_4)+x_2(y_3-y_1)+x_3(y_4-y_2)+x_4(y_1-y_3)\}.$$

3. Determine k in order that the points $(2, -1)$, $(-3, 4)$ and $(k, 5)$ may lie on a straight line. *Ans.* -4.

1·7. Polar co-ordinates. Another convenient way of fixing the position of a point on a plane is to take a fixed line OX, called the **initial line** through a fixed point O, called the **pole.** If the angle XOP and the distance OP are given, the position of the point P is known.



The length OP is called **radius vector,** and the angle XOP (positive if measured counter clockwise from OX) the **vectorial angle.** If $OP=r$, $\hat{XOP}=\theta$, the **polar co-ordinates** of P are (r, θ).

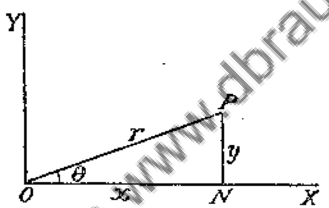
The polar co-ordinates are written in the same manner as the Cartesian Co-ordinates, the radius vector coming first.

The radius vector is considered positive if measured from the pole along the line bounding the vectorial angle.

A special feature of the polar co-ordinates is that the same point can be represented in an infinite number of ways. For example, the coordinates of P can be written as follows :

(r, θ), $(-r, \theta+\pi)$, $(-r, \theta-\pi)$, $(r, \theta-2\pi)$, and any set of co-ordinates obtained from these by going round the pole once, twice, etc., in the appropriate direction. Thus P will also be represented by $(r, \theta+2\pi)$, $(-r, \theta+3\pi)$, $(-r, \theta-3\pi)$, $(r, \theta 4-\pi)$ etc.

1·71. Relation between rectangular and polar co-ordinates.



The pole and the origin are the same point O. The x—axis is taken along the initial line OX and the y—axis a perpendicular at O. Join OP and draw PN perpendicular to OX.

Let (x, y) and (r, θ) be respectively the Cartesian and polar co-ordinates of P.

Then, $ON=x$, $PN=y$, $N\hat{O}P=\theta$, and $OP=r$.

From the rt.-angled triangle PON,

$$x=r\cos\theta,\ y=r\sin\theta.$$

Any relation between x and y can, therefore, be changed into a relation between r and θ by writing $r\cos\theta$ for x and $r\sin\theta$ for y.

Squaring and adding the corresponding sides of the above relationship,

$$x^2+y^2=r^2,\ \ i.e.,\ r=\sqrt{x^2+y^2},$$

and by division $\theta = \tan^{-1} \dfrac{y}{x}$,

which will transform a relation between r and θ into a relation between x and y.

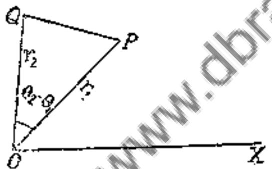
Ex. If x, y be related by means of the equation

$$(x^2+y^3-ax)^2 = a^2(x^2+y^2),$$

find the corresponding relation between r and θ.

Ans. $r^2 - 2ar\cos\theta - a^2\sin^2\theta = 0$.

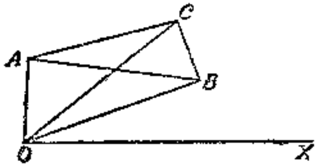
1·72. Distance between two points in polar co-ordinates.



Let (r_1, θ_1), (r_2, θ_2) be the polar co-ordinates of the two points P and Q. The angle POQ is obviously $\theta_2-\theta_1$, and from the triangle POQ,

$$PQ^2 = OP^2 + OQ^2 - 2OP.\,OQ\cos POQ$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1).$$

1·73. Area of a triangle in polar co-ordinates.



Let ABC be the triangle, the co-ordinates of the points A, B, C being (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) respectively.

$$\triangle ABC = \triangle OBC + \triangle OCA - \triangle AOB$$
$$= \tfrac{1}{2}OB.OC \sin BOC + \tfrac{1}{2}OC.OA \sin COA - \tfrac{1}{2} OB.OA \sin BOA.$$
$$= \tfrac{1}{2} r_2r_3 \sin(\theta_3 - \theta_2) + \tfrac{1}{2}r_3r_1 \sin(\theta_1 - \theta_3) - \tfrac{1}{2}r_2r_1 \sin(\theta_1 - \theta_2),$$

i.e., the area of the triangle ABC is

$$\tfrac{1}{2} \{r_2r_3 \sin(\theta_3 - \theta_2) + r_3r_1 \sin(\theta_1 - \theta_3) + r_1r_2 \sin(\theta_2 - \theta_1)\}.$$

Ex. Show that the area of the triangle whose vertices are

$$(c, \theta)\ (2c, \theta + \tfrac{1}{3}\pi),\ (3c, \theta + \tfrac{2}{3}\pi)$$

is

$$\tfrac{5}{4} \sqrt{3}\, c^2.$$

EXAMPLES ON CHAPTER I

1. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the middle points of the sides of a triangle, find the co-ordinates of its vertices.

2. Prove that the co-ordinates of the point of intersection of any two of the internal bisectors of the angles of a triangle whose vertices are (α_1, β_1), (α_2, β_2) and (α_3, β_3) are

$$\frac{a\alpha_1 + b\alpha_2 + c\alpha_3}{a+b+c} \quad \text{and} \quad \frac{a\beta_1 + b\beta_2 + c\beta_3}{a+b+c},$$

where a, b, c are the lengths of the sides of the triangle. Use this result to establish the concurrence of the internal bisectors.

3. Show that the points (6, 1), (7, 3), (9, 0) and (2, −2) are the angular points of a parallelogram.

4. If the point (x, y) be equidistant from the points (−1, 1) and (−2, −3), show that $2x + 8y + 11 = 0$, the axes being rectangular.

5. The cartesian co-ordinates (x, y) of a point on a curve are given by

$$x : y : 1 = t^3 : t^2 - 3 : t - 1,$$

where t is a parameter, show that the points given by $t=a$, b, c are collinear, if

$$abc-(bc+ca+ab)+3(a+b+c)=0$$

(*Math. Tripos 1946*)

6. The sides AB, BC, CA of a triangle ABC are bisected at D, E and F. Prove analytically that the triangle DEF is one-fourth of the triangle ABC.

CHAPTER II

LOCUS AND THE STRAIGHT LINE

2·1. Locus. The locus of a point is the path traced by the point when it moves in accordance with certain given conditions. Thus, for example, if a point moves such that it is always at the same distance from a fixed point O, it will lie on a circle, with centre O and radius equal to the given distance. This circle will be called the locus of the point.

2·11. Equation to a curve. The relation which exists between the co-ordinates of any point on a curve is called the equation to the curve. For example, if (x, y) be the co-ordinates of any point on a straight line which joins the two points (a_1, b_1), (a_2, b_2), its equation will be

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0,$$

for this is the relation between x and y obtained by considering the area of the triangle formed by these points.

It should be noted that the equation to a curve expresses the law governing all points which lie on that curve, and points not obeying that law will not satisfy the equation to the curve.

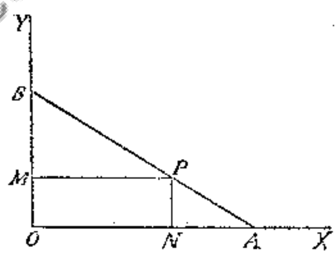
It is not always possible to express the relationship between the co-ordinates of any point on a locus in the form of an equation. For example, the locus of all points of which the x and y co-ordinates are always positive cannot be written as an equation. The same is true of points of which the distances from a fixed point are less than a given distance. The points in the latter case can be conceived of collectively as lying within the circumference of a fixed circle. It is thus obvious that a locus is not necessarily a line straight or curved. When, however, a locus belongs to this class its equation will represent a curve on which will lie only such points as are governed by the law of the locus.

Note 1. To obtain the equation to a curve or the locus of a point under given conditions, it is customary to assume the co-ordinates of any point on the locus as different from x, y which are treated as current co-ordinates. The co-ordinates of the point are changed to current co-ordinates after the given law has been expressed algebraically. This device suitable for the beginner may conveniently be ignored by the advanced student to whom the assumption of x and y as the co-ordinates of any point on the locus will not create any difficulty.

Note 2. The co-ordinate axes where not specified should be so chosen that the problem is solved with the minimum of calculation.

Example. 1. A straight line AB, of fixed length, slides between two perpendicular lines, OX and OY, in such a way that the point A always lies on OX and the point B on OY. Find the locus of the point P which divides AB into two parts PA and PB such that $PA=a$ and $PB=b$.

Take OX and OY as co-ordinate axes. Draw PN and PM perpendiculars to OX and OY and let (x, y) be the



co-ordinates of P. From the similar triangles PAN and PBM,

$$\frac{AN}{PM}=\frac{AP}{PB}$$

i.e.,
$$\frac{\sqrt{a^2-y^2}}{x}=\frac{a}{b}$$

or
$$a^2b^2=a^2x^2+b^2y^2,$$

which is the equation to the locus of P.

2. P and Q are two variable points on the axes of x and y respectively, such that $OP+OQ=a$; find the equation of the locus of the foot of the perpendicular from the origin on PQ.

Ans. $(x+y)(x^2+y^2)=axy.$

3. A point moves such that the difference of its distances from the points $(c, 0)$, $(-c, 0)$ is $2a$. Determine the equation of its locus.

Ans. $\dfrac{x^2}{a^2}-\dfrac{y^2}{c^2-a^2}=1.$

4. The points A and B have the two-dimensional rectangular cartesian co-ordinates $(1, 0)$ and $(-1, 0)$ respectively. Find equations for the loci of points P and Q which are such that

$$AP+BP=4$$
$$AQ-BQ=\pm 1.$$

Ans. $3x^2+4y^2=12;\ 12x^2-4y^2=3.$

2·2. The Straight Line. We shall show that the equation to a straight line is of the first degree and conversely that every equation of the first degree represents a straight line.

A straight line is uniquely determined if the co-ordinates of any two points lying on it are known.

Let (x_1, y_1), (x_2, y_2) be the co-ordinates of two fixed points on a straight line and let (x, y) be a variable point. The area of the triangle formed by these three points is zero. The relation between x and y or the equation to the locus of (x, y), *i.e.*, the straight line is therefore

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

i.e., $x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - y_1x_2 = 0$, which can be written as

$$Ax + By + C = 0,$$

where A, B and C are the coefficients of x, y and the constant term in the above equation.

This is an equation of the first degree and thus *to every straight line there corresponds only a first degree equation.*

Conversely, let $Ax + By + C = 0$ be the general equation of the first degree. This will be the locus of certain points. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three different points on the locus. These co-ordinates will therefore satisfy the equation to the locus.

That is,

$$Ax_1 + By_1 + C = 0,$$

$$Ax_2 + By_2 + C = 0,$$

and $$Ax_3 + By_3 + C = 0.$$

Eliminating A, B and C,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

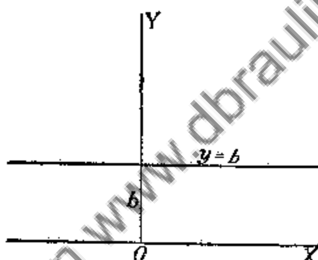
which expresses the fact that the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear. The locus is therefore a straight line.

Thus every equation of the first degree represents nothing but a straight line.

2·21. Particular Cases. It should be noted that the equation $Ax+By+C=0$ really involves two arbitrary constants, for it can be reduced to the form $px+qy+1=0$ on dividing by C. To evaluate the two unknowns p and q two equations are necessary, which will be obtained when two conditions satisfied by the line are given. Written in the general form the equation is said to have *two degrees of freedom*.

We shall give below a few particular cases.

Case 1. Straight lines drawn parallel to co-ordinate axes.



If a straight line is drawn parallel to the x-axis at a distance b from it, it will meet the x-axis at infinity. Two points on the line will be $(0, b)$ and (∞, b). To evaluate p and q in the general form

$$px+qy+1=0,$$

we notice that on putting $y=b$, $x=\infty$, q becomes infinite unless p is zero. Since the straight line does not lie wholly at infinity, p must be zero. Further since the straight line passes through $(0, b)$,

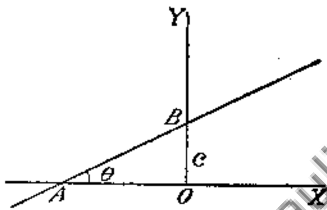
$$qb+1=0, \quad i.e., \quad q=-\frac{1}{b}.$$

Hence the equation to the desired line is $\mathbf{y=b}$.

This could be more easily obtained by noticing that the y-co-ordinate of any point on the line is b. Similarly the equation to a straight line drawn parallel to the axis of y

at a distance a from it is $\mathbf{x=a}$. The equations to the axes of x and y are respectively $\mathbf{y=0}$ and $\mathbf{x=0}$.

Case 2. Equation to a straight line which cuts off an intercept c on the y-axis and is inclined at an angle $\tan^{-1}$ m to the x-axis.



$\tan\theta = m$

Let the straight line cut the co-ordinate axes in A and B.

$$OA = OB\cot\theta = \frac{c}{m}.$$

The co-ordinates of A are therefore $\left(-\frac{c}{m}, 0\right)$. The co-ordinates of B are $(0, c)$.

Substituting in the general equation

$$px + qy + 1 = 0,$$

$$-\frac{c}{m}p + 1 = 0,$$

and $$qc + 1 = 0$$

i.e., $$p = \frac{m}{c}, \quad q = -\frac{1}{c}.$$

Hence the equation to the straight line is

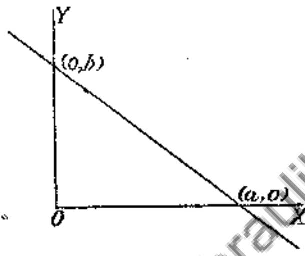
$$\frac{m}{c}x - \frac{1}{c}y + 1 = 0,$$

or $$\mathbf{y = mx + c}.$$

'm' is called the **slope** or **gradient** of the line.

Corollary. If a straight line passes through the origin, its equation is $y=mx$.

Case 3. Equation to a straight line which cuts off intercepts a and b from the axes of x and y respectively.



Two points on the straight line are obviously $(a, 0)$, $(0, b)$.

Substituting in the general equation $px+qy+1=0$,

$$ap+1=0,$$

and
$$bq+1=0$$

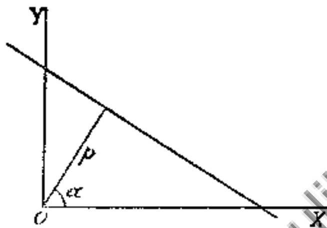
i.e.,
$$p=-\frac{1}{a},\ q=-\frac{1}{b}.$$

Hence the equation to the straight line is

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}.$$

In oblique co-ordinates too the equation to the straight line cutting off intercepts a and b from the co-ordinate axes is $\frac{x}{a}+\frac{y}{b}=1$. The reader should verify this by finding the locus of a movable point on the line. Thus the general equation of the first degree represents a straight line in oblique co-ordinates as well. It should now be easy to see that the condition of collinearity of three points is the same with oblique axes as with rectangular axes.

Case 4. Equation to a straight line such that the length of the perpendicular from the origin on it is p and the inclination of this perpendicular to the x-axis is α.



The intercepts which this line makes with the co-ordinate axes are obviously $p \sec \alpha$ and $p \operatorname{cosec} \alpha$. Proceeding as in case 3, the equation to the line will be seen to be

$$\mathbf{x \cos \alpha + y \sin \alpha = p}.$$

The reader should deduce the above cases by finding the locus of a moving point as an exercise.

Case 5. Equation to a straight line joining two points whose co-ordinates are $(\mathbf{x_1, y_1})$ and $(\mathbf{x_2, y_2})$.

The equation can be obtained from the general equation or independently in the form

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - y_1 x_2 = 0,$$

as shown in § 2·2.

This can be written as

$$x(y_1 - y_2) - y(x_1 - x_2) - x_1(y_1 - y_2) + y_1(x_1 - x_2) = 0,$$

or $$(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2)$$

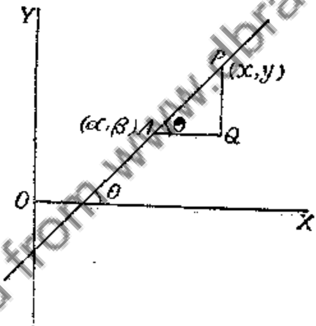
or $$\mathbf{y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)}.$$

The slope of this line is evidently $\dfrac{y_1 - y_2}{x_1 - x_2}$.

If (x_2, y_2) is arbitrary so is the factor $\frac{y_1-y_2}{x_1-x_2}$, and the equation to a straight line having one degree of freedom is $y-y_1=m(x-x_1)$. This equation therefore represents a straight line which passes through the point (x_1, y_1) and which can be made to satisfy *one* arbitrary condition.

We shall now deduce independently an important case of the equation to a straight line.

2·22. Equation to a straight line passing through a fixed point (α, β) and inclined at an angle θ to the x-axis.



Let P be any movable point (x, y) on the given line. Draw AQ, PQ parallel to the co-ordinate axes. Let the length AP be r.

In the right-angled triangle APQ,

$$AQ=AP\cos\theta,\ PQ=AP\sin\theta,$$

i.e.,
$$x-\alpha=r\cos\theta,\ y-\beta=r\sin\theta,$$

or
$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r.$$

The equation to the straight line is therefore

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}},$$

where l and m are respectively the cosine and sine of the angle which the line makes with the x-axis.

If the axes are *oblique* a straight line through (α, β) can still be expressed as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=r,$$

where l and m are not the cosine and sine of the angle which the line makes with the x-axis, but constants depending only on the direction of the line.

The student should note that distances measured on opposite sides of A will have opposite signs.

Corollary. The co-ordinates of a variable point on the line are given by

$$x=\alpha+lr, \quad y=\beta+mr.$$

Examples 1. A straight line drawn through the point (2, 1) is such that its point of intersection with the line $y-2x+6=0$ is at a distance $3\sqrt{2}$ from this point. Find the direction of the line.

If (x, y) is any point on the line,

$$x=2+lr, \quad y=1+mr.$$

If this lies also on the line

$$y-2x+3=0,$$

$$(1+mr)-2(2+lr)+6=0,$$

or $$r(m-2l)+3=0$$

But $$r=3\sqrt{2}, \text{ therefore,}$$

$$(m-2l)\sqrt{2}+1=0.$$

or $2\sqrt{2}\cos\theta-\sqrt{2}\sin\theta-1=0$, since l, m are $\cos\theta$ and $\sin\theta$ respectively.

Solving, $\theta=45°$.

2. Show that the equations of the straight lines passing through the point $(1, -1)$ and making angles of $150°$ and $30°$ respectively with the axis of x are

$$y+1 = \mp \frac{1}{\sqrt{3}}(x-1).$$

3. Find the equation of the line through the point $(2, 3)$ which makes equal intercepts on the axes.

Ans. $x+y=5$.

4. If p be the length of the perpendicular from the origin on the line $\frac{x}{a}+\frac{y}{b}=1$, show that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

5. A straight line is drawn through the point $(3, 4)$ to meet the curve $x^2+y^2=9$ in two points. Prove that the product of the distances of the points of intersection from the given point is constant for all positions of the line.

2·3. Point of Intersection of two straight lines.

Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ be the two straight lines and (x', y') their point of intersection.

Since (x', y') lies on both the lines,

$$a_1x'+b_1y'+c_1=0,$$

and

$$a_2x'+b_2y'+c_2=0.$$

Solving these as simultaneous equations,

$$x'=\frac{b_1c_2-c_1b_2}{a_1b_2-a_2b_1}, \quad y'=\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}.$$

As $a_1b_2-a_2b_1 \to 0$, both x' and $y' \to \infty$; the point of intersection therefore shifts to infinity. In this case therefore the lines will be parallel, as we shall see in § 2·4.

The student should note that as the denominator tends to zero, the numerator for x' or y' must remain finite if the lines are to remain distinct.

2·31. Equation to a Line passing through the point of intersection of two given lines.

Let $$a_1x+b_1y+c_1=0, \ldots\ldots \quad (1)$$

and $$a_2x+b_2y+c_2=0 \ldots\ldots \quad (2)$$

be the two given lines, and let

$$a_1 x+b_1 y+c_1+\lambda\ (a_2 x+b_2 y+c_2)=0 \ldots\ldots \quad (3)$$

be the equation to any other line, where λ is an arbitrary constant.

The values of x and y which satisfy (1) and (2) simultaneously obviously satisfy (3) also, which therefore is a straight line passing through the intersection of (1) and (2). This line will have one degree of freedom, that is, it can be made to satisfy only one arbitrary condition alone.

The student can find the point of intersection of (1) and (2) and write down the equation as in case 5 § 2·2; but written in form (3) the solutions of problems are obtained more conveniently.

Examples 1. Find the equation of the line joining the origin to the point of intersection of the lines

$$\frac{x}{3}+\frac{y}{4}=1 \text{ and } \frac{x}{4}+\frac{y}{3}=1.$$

Any line through the point of intersection of the given lines is $\frac{x}{3}+\frac{y}{4}-1+\lambda\left(\frac{x}{4}+\frac{y}{3}-1\right)=0.$

Since this passes through (0, 0), $-1-\lambda=0$ or $\lambda=-1$.

Hence the required line is

$$\frac{x}{3}+\frac{y}{4}-1-\left(\frac{x}{4}+\frac{y}{3}-1\right)=0, \text{ or } x=y.$$

2. Obtain the equations of the lines passing through the intersection of $4x-3y-1=0$ and $2x-5y+3=0$ and equally inclined to the axes. *Ans.* $x+y=2$; $x=y$

3. The base BC of a triangle ABC is bisected at the point (h, k) and the equations to the sides AB, AC are $hx+ky=1$ and $kx+hy=1$. Find the equation to the median through A.

Ans. $(2hk-1)(hx+ky-1)=(h^2+k^2-1)(kx+hy-1)$.

4. Find the area of the triangle made by the intersection of the lines

$$3x+4y=2$$
$$x+2y=3$$
$$3x+y=3$$

Ans. $\frac{529}{180}$ sq. units.

2·32. Equation to a straight line passing through a fixed point.

If the equation to a straight line can be written in the form

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0,$$

where a_1, b_1, c_1, a_2, b_2, c_2 are fixed and λ is a variable, the line passes through the point of intersection of the lines

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0,$$

which is a fixed point.

An equation of the first degree containing one arbitrary parameter represents a straight line passing through a fixed point.

Examples 1. A straight line moves so that the sum of the reciprocals of its intercepts on two fixed intersecting lines is constant; show that it passes through a fixed point.

Take the fixed intersecting lines as the co-ordinate axes and let the intercepts be a and b.

Then, $$\frac{1}{a}+\frac{1}{b}=k.$$

The equation to the line is

$$\frac{x}{a}+\frac{y}{b}=1.$$

Eliminating b,

$$\frac{x}{a}+y\left(k-\frac{1}{a}\right)=1,$$

Or $$ky-1+\frac{1}{a}\left(x-y\right)=0.$$

Since a is arbitrary, the line passes through the point of intersection of $x-y=0$, $ky-1=0$, *i.e.*, through $\left(\frac{1}{k}, \frac{1}{k}\right)$ which is a fixed point.

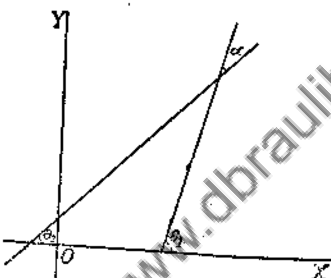
2. Given the vertical angle of a triangle in magnitude and position, and the sum of the reciprocals of the sides, show that the base will pass through a fixed point.

Hint. Take the sides of the triangle as co-ordinate axes.

3. A variable straight line divides one side of a given triangle internally and another externally in the same ratio. Prove that the join of the points of division passes through a fixed point. Interpret the result geometrically.

2·33. Straight Line at Infinity. We have seen that the equation $Ax+By+C=0$ represents a straight line. Writing this as $-\dfrac{x}{\frac{C}{A}}-\dfrac{y}{\frac{C}{B}}=1$, we see that the intercepts on the axes of x and y are respectively $-\dfrac{C}{A}$ and $-\dfrac{C}{B}$. If one of the quantities A or B tends to zero, C remaining finite, the straight line becomes parallel to one of the co-ordinate axes since the length of the intercept on that axis becomes infinite. If however both A and B tend to zero simultaneously, the lengths of the intercepts on both the axes tend to infinity. The equation which then reduces to $C=0$ represents a straight line, lying wholly at infinity.

2·4. Angle between two lines.



Case I. Let α be the angle between two straight lines whose inclinations to the x-axis are θ_1 and θ_2 and equations $y=m_1x+c_1$, $y=m_2x+c_2$ respectively.

Then, $\tan\theta_1=m_1$, $\tan\theta_2=m_2$.

Obviously, $\alpha=\theta_1-\theta_2$.

Hence, $\tan\alpha=\tan(\theta_1-\theta_2)$

$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$=\frac{m_1-m_2}{1+m_1m_2}$$

The angle between the straight lines whose slopes are m_1 and m_2 is therefore

$$\tan^{-1}\frac{\mathbf{m_1-m_2}}{\mathbf{1+m_1m_2}}.$$

Note. The student will sometimes find that his result for $\tan\alpha$ is negative. This will merely mean that instead of getting the acute angle of intersection he is getting its supplement which may as well be regarded as the angle of intersection of two straight lines. It is however better to disregard the sign and thus obtain only the acute value of α.

Cor. 1. *Two lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$ *are parallel if* $m_1=m_2$. For this gives a zero value to the angle of intersection which is the condition for two lines to be parallel.

Cor. 2. *Two lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$ *are perpendicular if* $m_1m_2=-1$, because this obviously makes the angle of intersection $90°$.

Note. If the equations to the two straight lines are

$$a_1x+b_1y+c_1=0,$$

and

$$a_2x+b_2y+c_2=0,$$

their slopes are $-\frac{a_1}{b_1}$ and $-\frac{a_2}{b_2}$ as can easily be seen by writing the equations in the form $y=mx+c$.

The **condition of parallelism** is therefore

$$-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$$

or

$$\mathbf{a_1b_2-a_2b_1=0},$$

and the **condition of perpendicularity** is

$$\left(-\frac{a_1}{b_1}\right).\left(-\frac{a_2}{b_2}\right)=-1$$

or

$$\mathbf{a_1a_2+b_1b_2=0}$$

Case 2. If the equations to the two lines are given as $x\cos\alpha_1+y\sin\alpha_1-p_1=0$ and $x\cos\alpha_2+y\sin\alpha_2-p_2=0$, the angle between them is $\alpha_1 \sim \alpha_2$ for the angle between two lines is the same as the angle between their perpendiculars. Two lines in this case will therefore be parallel if $\alpha_1=\alpha_2$, and perpendicular, if $\alpha_1 \sim \alpha_2=\frac{\pi}{2}$

2·41. Equations to straight lines drawn respectively parrallel and perpendicular to the line $ax+by+c=o$ *through the point* (x_1, y_1).

The slope of the given line is $-\frac{a}{b}$. Any line through (x_1, y_1) is

$$y-y_1=m(x-x_1),\dots\dots\dots(1)$$

where m is arbitrary.

This is parallel to the given line if $m=-\frac{a}{b}$.

Hence the required parallel line is

$$y-y_1=-\frac{a}{b}(x-x_1)$$

or $$a(x-x_1)+b(y-y_1)=0,$$

which may be obtained from the equation to the given line on replacing x *and* y *by* $x-x_1$, $y-y_1$ *respectively and omitting the constant term.*

Again, (1) is perpendicular to the given line if

$$m.\left(-\frac{a}{b}\right)=-1$$

or $$m=\frac{b}{a}.$$

The required perpendicular line is therefore

$$y-y_1=\frac{b}{a}(x-x_1),$$

or $$a(y-y_1)-b(x-x_1)=0,$$

which may be obtained from the equation to the given line on *replacing* x *by* $x-x_1$, y *by* $y-y_1$, *interchanging the co-efficients of* x *and* y, *changing the sign of one of the co-efficients and omitting the constant term.*

Ex. Find the equations to the diagonals of the parallelogram formed by the lines

$$lx+my+n=0, \quad l'x+m'y+n=0,$$
$$lx+my+n'=0, \quad l'x+m'y+n'=0,$$

and show that they are at right angles if $l^2+m^2=l'^2+m'^2$.

The straight line through the intersection of the first two lines is

$$lx+my+n+\lambda(l'x+m'y+n)=0 \qquad \ldots\ldots(1)$$

and the straight line through the intersection of the next two lines is

$$lx+my+n'+\mu(l'x+m'y+n')=0 \qquad \ldots\ldots(2)$$

where λ and μ are arbitrary. If the values of λ and μ can be so determined as to make (1) and (2) the same, either will represent the equation to one diagonal of the parallelogram. It is easy to see that $\lambda=\mu=-1$ reduces (1) and (2) to

$$(l-l')x+(m-m')y=0,$$

which is accordingly the equation to one diagonal.

The equation to the other diagonal will similarly be obtained by choosing λ' and μ' such that the equations

$$lx+my+n+\lambda'(l'x+m'y+n')=0,$$
$$\text{and} \quad lx+my+n'+\mu'(l'x+m'y+n)=0$$

become identical.

It is easy to see that this condition is satisfied if $\lambda'=\mu'=1$, and the equation to the other diagonal is therefore

$$(l+l')x+(m+m')y+n+n'=0.$$

The slopes of the two diagonals are respectively

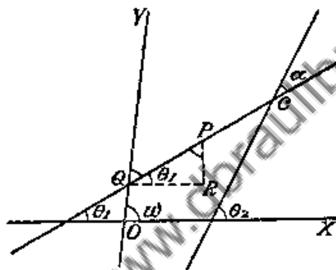
$$-\frac{l-l'}{m-m'} \text{ and } -\frac{l+l'}{m+m'}.$$

the angle between them is therefore a right angle if

$$\left(-\frac{l-l'}{m-m'}\right).\ \left(-\frac{l+l'}{m+m'}\right)=-1.$$

i.e., if $l^2+m^2=l'^2+m'^2$.

2·42. Angle between two lines when the axes are oblique.



If P be any point (x, y) on the line inclined at an angle θ_1 to the axis of x and cutting off an intercept c_1 from the y-axis, the triangle PRQ, where PR and QR are drawn parallel to OY and OX, gives

$$\frac{QR}{\sin QPR}=\frac{PR}{\sin PQR},$$

$$\text{or}\quad \frac{x}{\sin(\omega-\theta_1)}=\frac{y-c_1}{\sin\theta_1}$$

as the equation to the line under consideration.

This can be written as $y=m_1x+c_1$ where

$$\tan\theta_1=\frac{m_1\sin\omega}{1+m_1\cos\omega}.$$

If the equation to the line inclined at an angle θ_2 to the x-axis be $y=m_2x+c_2$, we similarly obtain

$$\tan\theta_2=\frac{m_2\sin\omega}{1+m_2\cos\omega}.$$

If α is the angle of intersection of the two lines,

$$\tan\alpha = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

Substituting for $\tan\theta_2$ and $\tan\theta_1$,

$$\alpha = \tan^{-1}\frac{(m_2 - m_1)\sin\omega}{1 + (m_1 + m_2)\cos\omega + m_1 m_2}$$

If $m_1 = m_2$ the lines are therefore parallel and if $1 + (m_1 + m_2)\cos\omega + m_1 m_2 = 0$, the lines are perpendicular.

Ex. 1. If the straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ make equal angles with the axis of x and be not parallel to one another, prove that $m_1 + m_2 + 2m_1 m_2 \cos\omega = 0$ where ω is the angle between the axes.

Ex. 2. The sides AB, BC, CD, DA of a quadrilateral have equations $x + 2y = 3$, $x = 1$, $x - 3y = 4$, $5x + y + 12 = 0$ respectively. Show that the diagonals AC and BD are at right angles.

[Entrance Scholarship Exam., Manchester, 1945.]

Ex. 3. Find the equation of a straight line which passes through the point of intersection of two given lines and is perpendicular to a third line in their plane.

Prove that the point $(-1, 4)$ is the orthocentre of the triangle which is formed by the lines whose equations are

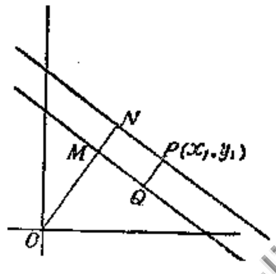
$$x - y + 1 = 0,\ x - 2y + 4 = 0,\ 9x - 3y + 1 = 0.$$

[Math. Tripos 1912.]

Ex. 4. The three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$, $2x - y - 4 = 0$ form the three sides of two squares. Find the equation to the fourth side of each square.

Ans. $y - 2x = 6$; $y - 2x = -14$.

2·5. Length of the Perpendicular from a fixed point on a given straight line.



Let the given line be

$$x \cos \alpha + y \sin \alpha - p = 0, \quad \ldots\ldots\ldots(1)$$

and let the fixed point be P with co-ordinates (x_1, y_1). The perp. OM from the origin on the given line is thus p.

The equation to a line parallel to (1) through P is

$$(x - x_1) \cos \alpha + (y - y_1) \sin \alpha = 0$$

or $$x \cos \alpha + y \sin \alpha - (x_1 \cos \alpha + y_1 \sin \alpha) = 0.$$

The perp. ON from the origin on this is therefore

$$x_1 \cos \alpha + y_1 \sin \alpha.$$

If PQ is the perp. from P on (1), from the rectangle $PQMN$,

$$PQ = MN$$

But $$MN = ON - OM$$

$$= x_1 \cos \alpha + y_1 \sin \alpha - p.$$

Hence the length of the perpendicular from (x_1, y_1) *on the line* $x \cos \alpha + y \sin \alpha - p = 0$ *is*

$$\mathbf{x_1 \cos \alpha + y_1 \sin \alpha - p.}$$

If the equation to the line is

$$Ax + By + C = 0,$$

we can write it as

$$\frac{A}{\sqrt{(A^2+B^2)}}x+\frac{B}{\sqrt{(A^2+B^2)}}y+\frac{C}{\sqrt{(A^2+B^2)}}=0.$$

Putting $\frac{A}{\sqrt{(A^2+B^2)}}=\cos\theta$, $\frac{B}{\sqrt{(A^2+B^2)}}=\sin\theta$, and

$$\frac{C}{\sqrt{(A^2+B^2)}}=-P,$$

we see that it reduces to the form (1), and therefore the length of the perpendicular from the point (x_1, y_1) on the line $Ax+By+C=0$ is

$$x_1\cos\theta+y_1\sin\theta-P,$$

or, substituting for $\cos\theta$, $\sin\theta$ and P,

$$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}.$$

If the given point is the origin, this reduces to

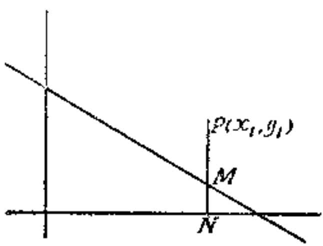
$$\frac{C}{\sqrt{(A^2+B^2)}}.$$

The student will find that the value of the expression

$$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}$$

in numerical problems is sometimes positive, sometimes negative. He should disregard the negative sign and give the result as positive unless there is some special reason for not doing so.

2·6. Condition that the point (x_1, y_1) may lie on one side or the other of the straight line $Ax+By+C=0$.



Let P be the point (x_1, y_1).

Let PN be the ordinate through P meeting the given line in M.

PM will be positive for points on one side of the line and negative for points on the other side, for the lengths PM will be oppositely directed in the two cases.

The abscissa of M will be x_1, and since M lies on the straight line $Ax+By+C=0$, its ordinate y' will be given by

$$Ax_1+By'+C=0.$$

Hence,

$$y'=-\frac{Ax_1+C}{B}.$$

Therefore

$$\begin{aligned}PM&=PN-MN\\&=y_1-y'\\&=y_1+\frac{Ax_1+C}{B}\\&=\frac{Ax_1+By_1+C}{B}\end{aligned}$$

The point (x_1, y_1) thus lies on one side or the other of the straight line $Ax+By+C=0$ according as the expression Ax_1+By_1+C is positive or negative.

If the expression Ax_1+By_1+C be *positive* the point (x_1, y_1) is said to lie on the positive side of the line $Ax+By+C=0$, and if it be *negative*, the point is said to lie on the *negative side*.

The same point can be on the positive side of a straight line and on the negative side of the same straight line depending upon how the equation to the straight line is written. For example, the origin is on the positive side of the line $Ax+By+C=0$. But if the equation to the line be written as $-Ax-By-C=0$, the side on which the origin lies becomes negative. The student should try to explain this.

Corollary. Two points (x_1, y_1), (x_2, y_2) lie on the same side of the line $Ax+By+C=0$, if the expressions Ax_1+By_1+C and Ax_2+By_2+C have the same sign. But if these expressions have opposite signs, the points lie on opposite sides of the line.

Ex. 1. Show that the perpendicular from the origin upon the straight line joining the points $(c \cos \alpha, c \sin \alpha)$ and $(c \cos \beta, c \sin \beta)$ bisects the distance between them.

Hint. The points are equidistant from the origin.

Ex. 2. If p and p' be the perpendiculars from the origin upon the straight lines whose equations are

$$x \sec \theta + y \operatorname{cosec} \theta = a \text{ and } x \cos \theta - y \sin \theta = a \cos 2\theta,$$

prove that $4p^2 + p'^2 = a^2$.

Ex. 3. Prove that the product of the perpendiculars let fall from the points $(\pm \sqrt{a^2-b^2}, 0)$ upon the line $\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$ is b^2.

Ex. 4. Find the equations of the two straight lines drawn through the point $(0, a)$ on which the perpendiculars let fall from the point $(2a, 2a)$ are each of length a.

Find also the equation of the straight line joining the feet of these perpendiculars.

Ans. $y=a$, $4x-3y+3a=0$; $2x+y=5a$.

Ex. 5. Find the area of the parallogram the equations to whose sides are

$$l_1x+m_1y+n_1=0, \qquad l_2x+m_2y+n_2=0,$$

$$l_1x+m_1y+n_3=0, \qquad l_2x+m_2y+n_4=0.$$

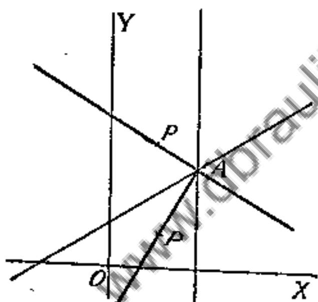
Ans. $\dfrac{(n_3-n_1)(n_4-n_2)}{l_1m_2-m_1l_2}$.

Ex. 6. Find the distance between the straight line $2x+y=3$ and a parallel to it through the point $(-1, -2)$.

Determine also the location of this point with respect to the given line. *Ans.* $\frac{7}{\sqrt{5}}$.

Ex. 7. Prove that the triangle formed by joining the points whose co-ordinates are (1, 2), (−2, 4), (3, −1) lies wholly on the positive side of the line $x+y=1$.

2·7. Straight lines bisecting the angles between two given straight lines.



Let the equations to the two given lines be

$$a_1x+b_1y+c_1=0 \quad \ldots \quad \ldots \quad (1),$$

and

$$a_2x+b_2y+c_2=0 \quad \ldots \quad \ldots \quad (2),$$

and let P be any point (x', y') on the internal or external bisector. The length of the perpendicular from P on (1) and (2) will have the same numerical value.

Let the equations (1) and (2) be so written that c_1 and c_2 have the same sign. The origin and the point P lie on the same side of each line provided P moves on the bisector of the angle containing the origin. If, however, P moves on the other bisector, the point P and the origin lie on the same side of one line and on opposite sides of the other line.

The lengths of the perpendiculars from (x', y') on (1) and (2) are respectively

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}} \text{ and } \frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Further, since the perpendiculars from the origin upon (1) and (2) will have the same signs as c_1 and c_2 respectively, the locus of P when it lies on the *bisector of the angle containing the origin will be*

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}}.$$

The equation to the *bisector of the angle not containing the origin will be*

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=-\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}}.$$

The important point to be remembered is that the equations to the given straight lines should be so written that the constant terms in them are both either positive or negative.

The student should verify that the two bisectors are at right angle to each other.

Ex. 1. Find the equations to the bisectors of the angles between the lines $3x+4y-1=0$, and $12x+5y+2=0$.

Writing the equations such that the constant terms have the same signs,

$$3x+4y-1=0,$$

and
$$-12x-5y-2=0.$$

The equation to the bisector of the angle in which the origin lies is

$$\frac{3x+4y-1}{\sqrt{3^2+4^2}}=\frac{-12x-5y-2}{\sqrt{12^2+5^2}},$$

i.e.,
$$99x+77y-3=0.$$

The equation to the bisector of the other angle is

$$\frac{3x+4y-1}{\sqrt{3^2+4^2}}=-\frac{-12x-5y-2}{\sqrt{12^2+5^2}},$$

i.e., $21x-27y+23=0.$

Ex. 2. Find the centre of the inscribed circle of the triangle the equations of whose sides are $4y+3x=0$, $12y-5x=0$ and $y-15=0$. *Ans.* $(1, 8)$.

2·8. Concurrence of three straight lines.

Let the equations to three given lines be

$$a_1 x+b_1 y+c_1=0 \qquad \dots \quad (1),$$

$$a_2 x+b_2 y+c_2=0 \qquad \dots \quad (2),$$

and

$$a_3 x+b_3 y+c_3=0 \qquad \dots \quad (3),$$

Let (x_1, y_1) be the common point of intersection of the lines.

Then,

$$a_1 x_1+b_1 y_1+c_1=0,$$

$$a_2 x_1+b_2 y_1+c_2=0,$$

and

$$a_3 x_1+b_3 y_1+c_3=0.$$

Eliminating x_1 and y_1,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0,$$

which is the condition for the concurrence of the given lines.

There is yet another test, very effective at times, to study the concurrence of three given lines. *If three constants* λ, μ, ν *can be so determined that*

$$\lambda\ (a_1 x + b_1 y + c_1) + \mu\ (a_2 x + b_2 y + c_2) + \nu\ (a_3 x + b_3 y + c_3) \qquad \ldots\ldots(4)$$

vanishes identically, the lines (1), (2) *and* (3) *will meet in a point.*

The co-ordinates of the point of intersection of any two of the given lines, say (2) and (3), will make the expressions $a_2 x + b_2 y + c_2$ and $a_3 x + b_3 y + c_3$ separately zero. Since the expression (4) vanishes for all values of x and y, the co-ordinates of the point of intersection of (2) and (3) will also satisfy the equation

$$a_1 x + b_1 y + c_1 = 0,$$

i. e., the three lines will be concurrent.

Ex. 1. Show that the lines $2x + y - 1 = 0$, $4x + 3y - 3 = 0$, and $3x + 2y - 2 = 0$ are concurrent.

Ex. 2. Find the value of k for which the lines $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$, and $4x + 5y + k = 0$ are concurrent. *Ans.* -45.

Ex. 3. In any triangle prove the concurrence of the following :

(1) the medians

(2) the altitudes

(3) the bisectors of the angles, either all internal or two internal and one external

(4) the perpendicular bisectors of the sides.

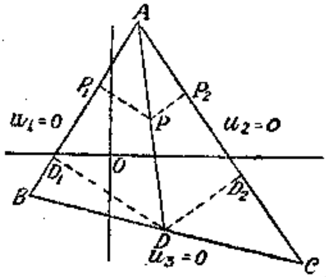
Let the sides of the triangle be

$$u_1 \equiv x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0,$$

$$u_2 \equiv x \cos \alpha_2 + y \sin \alpha_2 - p_2 = 0,$$

and $$u_3 \equiv x \cos \alpha_3 + y \sin \alpha_3 - p_3 = 0.$$

Let P be any point (x, y) on the median through the intersection of $u_1 = 0$, $u_2 = 0$, The perp. PP_1 is u_1; the perp. PP_2 is u_2.



But
$$\frac{PP_1}{PP_2}=\frac{DD_1}{DD_2}=\frac{\sin B}{\sin C},$$

where DD_1 and DD_2 are perpendiculars from D upon $u_1=0$ and $u_2=0$.

The locus of P or the equation to AD is therefore

$$u_1 \sin C - u_2 \sin B = 0. \quad \dots \quad (1)$$

The equations to the other two medians are similarly

$$u_2 \sin B - u_3 \sin A = 0 \quad \dots \quad (2),$$

and
$$u_3 \sin A - u_1 \sin C = 0 \quad \dots \quad (3).$$

Adding (1), (2) and (3) the left hand sides vanish identically. Hence the medians are concurrent.

The student should establish the other results in a like manner.

2·9. Equation to a Straight Line in Polar Co-ordinates. The general equation to a straight line in Cartesian co-ordinates is $px+qy=1$. Converting this into polar co-ordinates,

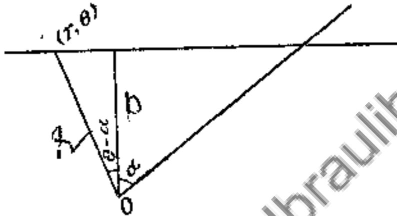
$$pr \cos\theta + qr \sin\theta = 1,$$

or
$$\frac{1}{r}=p\cos\theta+q\sin\theta,$$

which in polar co-ordinates is the most general equation to a straight line.

We shall obtain from first principles the equation to a straight line in the following cases.

Case I. If p is the length of the perpendicular from the pole upon a given straight line, α the angle which this

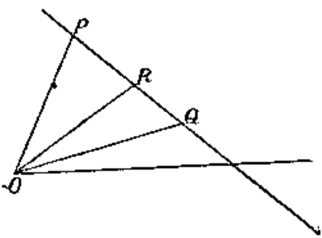


perpendicular makes with the initial line and (r, θ) the co-ordinates of any movable point on the line, then, obviously,

$$p = r \cos(\theta - \alpha),$$

which is the equation to the given line.

Case II. Let us obtain the equation to a straight line joining two given points.



Let Q and R be the given points with co-ordinates (r_1, θ_1), (r_2, θ_2) on the line QR. Let P be any point on this line.

Since $\triangle OQP = \triangle OQR + \triangle ORP$,

$$\tfrac{1}{2} r r_1 \sin(\theta - \theta_1) = \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \tfrac{1}{2} r r_2 \sin(\theta - \theta_2),$$

or $r_1 r_2 \sin(\theta_2 - \theta_1) + r r_2 \sin(\theta - \theta_2) + r r_1 \sin(\theta_1 - \theta) = 0$,

i.e., $$\frac{\sin(\theta_2 - \theta_1)}{r} + \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2} = 0,$$

which is the required equation.

Ex. 1. Show that the equation to the straight line passing through the pole and making an angle α with the initial line is $\theta = \alpha$.

Ex. 2. Show that the equation to the straight line perpendicular to the line $\frac{l}{r} = \cos(\theta - \alpha) + e \cos\theta$ is $\frac{l'}{r} = -\sin(\theta - \alpha) - e \sin\theta$. Obtain also the equation to the parallel through the pole.

Ans. $\theta = \tan^{-1} \frac{\cos\alpha + e}{\sin\alpha}$.

Ex. 3. Find the polar co-ordinates of the foot of the perpendicular from the pole on the line joining the two points (r_1, θ_1) and (r_2, θ_2).

Ans. $r_1 r_2 \sin(\theta_1 - \theta_2) / \sqrt{\{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2)\}}$,

$$\tan^{-1} \frac{r_2 \cos\theta_2 - r_1 \cos\theta_1}{r_1 \sin\theta_1 - r_2 \sin\theta_2}.$$

Ex. 4. Show that the equation to the straight line passing through the point of intersection of the lines

$\frac{l}{r} = A \cos\theta + B \sin\theta$ and $\frac{l'}{r} = A' \cos\theta + B' \sin\theta$ is

$$\frac{l + \lambda l'}{r} = (A + \lambda A') \cos\theta + (B + \lambda B') \sin\theta$$

EXAMPLES ON CHAPTER II

1. A stick of length l rests against the floor, and a wall of a room. If the stick begins to slide on the floor, find the locus of its middle point.

2. Show that the equations of the lines which pass through (x_1, y_1) and are inclined at an angle α with the line $ax+by+c=0$ are

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a\sin\alpha \pm b\cos\alpha & b\sin\alpha \pm a\cos\alpha & 0 \end{vmatrix} = 0$$

3. Prove that the locus of the foot of the perpendicular from the origin upon the line joining the points $(a\cos\varphi, b\sin\varphi)$ and $(-a\sin\varphi, b\cos\varphi)$, φ varying in any manner, is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

4. Through a fixed point (β, γ) lines P_1Q_1 and P_2Q_2 are drawn intersecting the axes of co-ordinates respectively in P_1, Q_1 and P_2, Q_2 and making angles θ_1 and θ_2 with the axis of x. Prove that the condition that P_1Q_2 may be parallel to P_2Q_1 is

$$\tan\theta_1 \tan\theta_2=\frac{\gamma^2}{\beta^2}.$$

[Math. Tripos, 1909.]

5. If a straight line passes through a fixed point, find the locus of the middle point of the portion of it which is intercepted between two given lines.

6. A variable straight line cuts off from n given concurrent straight lines intercepts, the sum of the reciprocals of which is constant. Show that it always passes through a fixed point.

Hint. Use polar co-ordinates.

Since $\triangle OQP = \triangle OQR + \triangle ORP$,

$$\tfrac{1}{2} r r_1 \sin(\theta - \theta_1) = \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \tfrac{1}{2} r r_2 \sin(\theta - \theta_2),$$

or $r_1 r_2 \sin(\theta_2 - \theta_1) + r r_2 \sin(\theta - \theta_2) + r r_1 \sin(\theta_1 - \theta) = 0$,

i.e., $$\frac{\sin(\theta_2 - \theta_1)}{r} + \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2} = 0,$$

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Ans. $r_1 r_2 \sin(\theta_1 - \theta_2) / \sqrt{\{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2)\}}$,

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$\frac{l}{r} = A\cos\theta + B\sin\theta$ and $\frac{l'}{r} = A'\cos\theta + B'\sin\theta$ is

$$\frac{l + \lambda l'}{r} = (A + \lambda A')\cos\theta + (B + \lambda B')\sin\theta$$

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1. A stick of length l rests against the floor, and a wall of a room. If the stick begins to slide on the floor, find the locus of its middle point.

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$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a\sin\alpha \pm b\cos\alpha & b\sin\alpha \pm a\cos\alpha & 0 \end{vmatrix} = 0$$

3. Prove that the locus of the foot of the perpendicular from the origin upon the line joining the points $(a\cos\varphi, b\sin\varphi)$ and $(-a\sin\varphi, b\cos\varphi)$, φ varying in any manner, is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

4. Through a fixed point (β, γ) lines P_1Q_1 and P_2Q_2 are drawn intersecting the axes of co-ordinates respectively in P_1, Q_1 and P_2, Q_2 and making angles θ_1 and θ_2 with the axis of x. Prove that the condition that P_1Q_2 may be parallel to P_2Q_1 is

$$\tan\theta_1 \tan\theta_2 = \frac{\gamma^2}{\beta^2}.$$

[Math. Tripos, 1909.]

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6. A variable straight line cuts off from n given concurrent straight lines intercepts, the sum of the reciprocals of which is constant. Show that it always passes through a fixed point.

Hint. Use polar co-ordinates.

20. Given n straight lines and a fixed point O; through O is drawn a straight line meeting these lines in the points $R_1, R_2, R_3, \ldots\ldots\ldots R_n$, and on it is taken a point R such that

$$\frac{n}{OR} = \sum_{p=1}^{n} \frac{1}{OR_p},$$

show that the locus of R is a straight line.

21. The Cartesian equations of the sides BC, CA, AB of a triangle are

$$u_1 \equiv a_1x + b_1y + c_1 = 0, \quad u_2 \equiv a_2x + b_2y + c_2 = 0,$$

$$u_3 \equiv a_3x + b_3y + c_3 = 0,$$

and a line is drawn through A parallel to BC, prove that its equation is

$$\frac{u_3}{a_3b_1 - a_1b_3} + \frac{u_2}{a_1b_2 - a_2b_1} = 0.$$

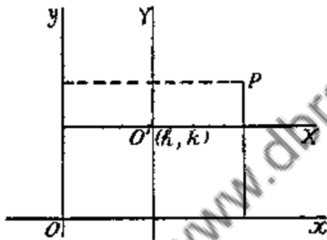
Show also that the equation of the line through A bisecting the side BC is

$$\frac{u_3}{a_3b_1 - a_1b_3} - \frac{u_2}{a_1b_2 - a_2b_1} = 0.$$

CHAPTER III

CHANGE OF AXES. INVARIANTS.

3·1. Change of Origin. Let (x, y) be the co-ordinates of any point P referred to rectangular axes Ox and Oy.



Let O' be the point (h, k) and $O'X$ and $O'Y$ lines drawn parallel to Ox and Oy respectively. Let (X, Y) be the co-ordinates of P referred to $O'X$ and $O'Y$ as co-ordinate axes. If then straight lines are drawn parallel to these axes, it is easily seen that

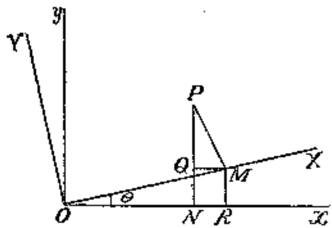
$$x = X + h, \quad y = Y + k.$$

If the locus of P with respect to Ox and Oy be $f(x, y) = 0$, the equation to the locus when the origin is transferred to O', the axes retaining their directions, becomes $f(X+h, Y+k) = 0$ where X, Y are the current co-ordinates with reference to the new axes.

The same law of transformation will apply in the case of oblique axes also since it is independent of the angle between the axes. The proof is too simple to be given.

It may be easily seen that this transformation affects only the first degree terms in an equation.

3·2. Change of rectangular axes without change of origin.



Let Ox, Oy be changed to the new axes OX, OY where θ is the angle xOX or yOY. Let the co-ordinates of any point P be (x, y) referred to old axes and (X, Y) referred to new axes.

Draw PN, PM perpendicular to Ox, and OX and MR, MQ perpendicular to Ox, and PN.

Then, $ON=x$, $PN=y$, $OM=X$, $PM=Y$.

Also, $\angle QPM=\theta$.

In the triangle QPM,

$$PQ=PM\cos\theta.$$

But, $PQ=PN-NQ=PN-MR=y-X\sin\theta$

Hence,

$$y-X\sin\theta=Y\cos\theta$$

i.e. $$\mathbf{y=X\sin\theta+Y\cos\theta} \quad \dots \quad (1)$$

Also, $OM\cos\theta=OR=ON+NR=x+QM=x+Y\sin\theta$.

Hence, $$\mathbf{x=X\cos\theta-Y\sin\theta} \quad \dots \quad (2)$$

From (1) and (2), or independently,

$$X=\quad x\cos\theta+y\sin\theta$$

$$Y=-x\sin\theta+y\cos\theta.$$

The equation $f(x, y)=0$ is transformed to

$$f(X \cos\theta - Y \sin\theta,\ X \sin\theta + Y \cos\theta)$$

when the axes are rotated through an angle θ without change of origin.

If the origin be transferred to (h, k) and the axes turned through an angle θ, the transformation will be given by the relations

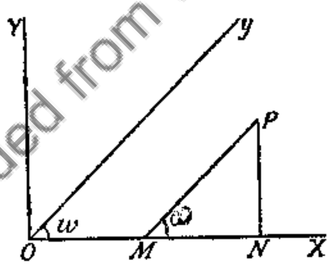
$$x = h + X \cos\theta - Y \sin\theta,\ y = k + X \sin\theta + Y \cos\theta.$$

Ex. The equation to the locus of a point is

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0.$$

Show that if the origin be transferred to (2, 3) and the axes rotated through an angle $\tan^{-1} 2$, the equation referred to new axes becomes $\dfrac{X^2}{6} + \dfrac{Y^2}{4} = 1.$

3·3. Transformation of oblique axes to rectangular axes with the same origin and same abscissa.



Let the co-ordinates of a point P be (x, y) referred to oblique axes OX, Oy, and (X, Y) referred to rectangular axes OX, OY.

If PM is drawn parallel to Oy and PN parallel to OY, then obviously,

$$X = x + Y \cot\omega$$

i. e., $x = X - Y \cot \omega$,

and $y = Y \operatorname{cosec} \omega$.

Ex. 1. Obtain the length of the perpendicular from (x_1, y_1) upon the line $Ax + By + C = 0$, the axes being inclined at an angle ω.

Transforming from oblique axes to rectangular axes with the same origin and same abscissa, the equation to the given line becomes

$$A(X - Y \cot \omega) + BY \operatorname{cosec} \omega + C = 0,$$

or $$A \sin \omega . X + Y(B - A \cos \omega) + C \sin \omega = 0.$$

The perpendicular from (x_1', y_1') on this is (§ 2·5)

$$\frac{A \sin \omega . x_1' + (B - A \cos \omega)\, y_1' + C \sin \omega}{\sqrt{[A^2 \sin^2 \omega + (B - A \cos \omega)^2]}} \quad ...(1)$$

Since (x_1', y_1') are the co-ordinates of (x_1, y_1) referred to new axes,

$$x_1 = x_1' - y_1' \cot \omega,$$

and $$y_1 = y_1' \operatorname{cosec} \omega$$

i. e., $x_1' = x_1 + y_1 \cos \omega$ and $y_1' = y_1 \sin \omega$.

Substituting in (1), the numerator becomes

$A \sin \omega . (x_1 + y_1 \cos \omega) + (B - A \cos \omega)\, y_1 \sin \omega + C \sin \omega$,

i. e., $(A x_1 + By_1 + C) \sin \omega$.

Hence the required perpendicular length is

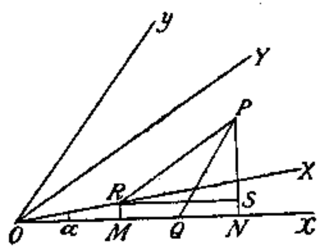
$$\frac{(A x_1 + B y_1 + C) \sin \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)}}.$$

Ex. 2. From a given point (h, k) perpendiculars are drawn to the axes and their feet are joined, prove that the length of the perpendicular from (h, k) upon this line is

$$\frac{h k \sin^2 \omega}{\sqrt{(h^2 + k^2 + 2hk \cos \omega)}},$$

and that its equation is $hx - ky = h^2 - k^2$, ω being the angle between the axes.

3·4. Transformation of one set of oblique axes to another with the same origin.



Let ω, ω′ be the angles between Ox, Oy and OX, OY respectively and α the inclination of the new x—axis to the old. Let the co-ordinates of any point P be (x, y) and (X, Y) referred to old and new axes respectively.

Draw PQ parallel to Oy and PR parallel to OY. Then $OQ=x$, $PQ=y$ and $OR=X$, $PR=Y$.

Let PN and RM be drawn perpendicular to Ox and RS perpendicular to PN.

In the $\triangle^s$ PQN, PRS, $\angle PQN=\omega$, $\angle PRS=\omega'+\alpha$.

$$\text{Therefore, } y\sin\omega=PN=PS+SN=PS+RM$$
$$=Y\sin(\omega'+\alpha)+X\sin\alpha,$$

$$\text{i.e.,}\qquad y=X\frac{\sin\alpha}{\sin\omega}+Y\frac{\sin(\alpha+\omega')}{\sin\omega}\qquad\ldots\quad(1).$$

Further, since the inclinations of OX and OY to Oy are $\omega-\alpha$ and $\omega-(\alpha+\omega')$, we obtain as in (1),

$$x\sin\omega=X\sin(\omega-\alpha)+Y\sin\{\omega-(\alpha+\omega')\},$$

$$\text{i.e.,}\qquad x=X\frac{\sin(\omega-\alpha)}{\sin\omega}+Y\frac{\sin\{\omega-(\alpha+\omega')\}}{\sin\omega}\ldots\quad(2)$$

The equation $f(x, y)=0$ thus transforms into

$$f\left(X\frac{\sin(\omega-\alpha)}{\sin\omega}+Y\frac{\sin(\omega-\beta)}{\sin\omega}, X\frac{\sin\alpha}{\sin\omega}+Y\frac{\sin\beta}{\sin\omega}\right),$$

where α and β are the inclinations of the new axes of x and y to the old x-axis.

It should be noted that the equations of transformation in the case of oblique and rectangular axes, the origin remaining the same, have the same form, namely,

$$x=\lambda X+\mu Y, \quad y=\lambda' X+\mu' Y,$$

where $\lambda, \mu, \lambda', \mu'$ are constants depending upon the conditions of transformation.

3·5. The degree of an equation and change of axes. In the above we have seen that the relation between the old and new co-ordinates in a general transformation of the axes is of the form

$$x=pX+qY+r \text{ and } y=p'X+r'Y+r'.$$

The expressions on the right hand sides are of the first degree and if they replace x and y in any given equation, the degree of the equation will *not* be raised. The degree of the equation will not be lowered either, for, if it were so, the transition from the new axes to the old will result in an equation in x and y of which the degree will be lower than that of the original equation in x and y.

The transformation of co-ordinate axes therefore does not affect the degree of an equation.

3·6. Removal of xy term from $ax^2+2hxy+by^2$ in the case of rectangular axes. Since a change of origin alone does not affect the second or higher degree terms in an equation we shall assume the origin to be fixed and rotate the axes through an angle θ. If X, Y are the new co-ordinates,

$$x = X\cos\theta - Y\sin\theta,$$

and
$$y = X\sin\theta + Y\cos\theta.$$

Substituting in the expression $ax^2+2hxy+by^2$,

$$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2,$$

which can be written as

$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)X^2 - 2\{(a-b)\sin\theta\cos\theta - h(\cos^2\theta - \sin^2\theta)\}XY + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)Y^2.$$

The term in XY disappears if

$$(a-b)\sin\theta\cos\theta - h\cos 2\theta = 0,$$

i.e., if
$$\tan 2\theta = \frac{2h}{a-b}.$$

Since a real value of θ is always possible, the axes can be so chosen as to reduce the expression $ax^2+2hxy+by^2$ to the form AX^2+BY^2.

If the expression $ax^2+2hxy+by^2$ is a perfect square, *i.e.*, if $h^2=ab$, rotation through the above angle will also make either the coefficient of $X^2=0$, or the coefficient of $Y^2=0$. The proof is left as an exercise to the reader.

3·7. Invariants. If any change of axes transforms $ax^2+2hxy+by^2$ into $a'X^2+2h'XY+b'Y^2$, then

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

and
$$\frac{ab-h^2}{\sin^2\omega} = \frac{a'b'-h'^2}{\sin^2\omega'},$$

where ω and ω' are the angles of inclination of the old and new sets of axes.

Obviously the origin will be the same for the two sets of axes. Let (x, y) be the co-ordinates of any point P referred to old axes and (X, Y) the co-ordinates of the same point referred to new axes.

Since the distance OP is independent of the choice of axes,

$x^2+y^2+2xy\cos\omega$ is changed into $X^2+Y^2+2XY\cos\omega'$.

Also, by supposition,

$ax^2+2hxy+by^2$ is changed into $a'X^2+2h'XY+b'Y^2$.

Therefore, $ax^2+2hxy+by^2+\lambda(x^2+y^2+2xy\cos\omega)$ will be changed into

$$a'X^2+2h'XY+b'Y^2+\lambda(X^2+Y^2+2XY\cos\omega'),$$

where λ is any constant.

Since the transformation relation is of the form $x=\lambda X+\mu Y$, $y=\lambda' X+\mu' Y$, any value of λ which makes one of the above expressions a perfect square will also make the other a perfect square*.

The first expression is a perfect square if

$$(a+\lambda)(b+\lambda)-(h+\lambda\cos\omega)^2=0 \qquad \ldots (1),$$

and the second is a perfect square if

$$(a'+\lambda)(b'+\lambda)-(h'+\lambda\cos\omega')^2=0 \qquad \ldots (2).$$

Equations (1) and (2) in λ must be the same.

Rearranging terms in (1) and (2),

$$\lambda^2\sin^2\omega+(a+b-2h\cos\omega)\lambda+ab-h^2=0, \qquad \ldots (3),$$

and $\lambda^2\sin^2\omega'+(a'+b'-2h'\cos\omega')\lambda+a'b'-h'^2=0 \qquad \ldots (4)$

*If the first expression as a perfect square is $(Px+Qy)^2$, it will be changed into $\{P(\lambda X+\mu Y)+Q(\lambda' X+\mu' Y)\}^2$ *i.e.*, into $(P'X+Q'Y)^2$ which is also a perfect square.

Comparing coefficients in (3) and (4),

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'} \quad \ldots \quad (5),$$

and $$\frac{ab-h^2}{\sin^2\omega} = \frac{a'b'-h'^2}{\sin^2\omega'} \quad \ldots \quad (6).$$

The expressions $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$ and $\dfrac{ab-h^2}{\sin^2\omega}$ are called **invariants** for their values are independent of the transformation of axes.

Corollary. *If any change of rectangular axes converts* $ax^2+2hxy+by^2$ *into* $a'X^2+2h'XY+b'Y^2$, *then will*

$$a+b=a'+b',$$

and $$ab-h^2=a'b'-h'^2.$$

This follows from (5) and (6) on putting $\omega=\omega'=\pi/2$.

EXAMPLES ON CHAPTER III

1. The co-ordinates of a point referred to two sets of rectangular axes with the same origin are (x, y) and (X, Y). If $ux+vy$ where u and v are independent of x and y becomes $UX+VY$, then

$$u^2+v^2=U^2+V^2.$$

2. If one system of oblique axes be transformed to another system according to the formula

$$x=mx'+ny', \quad y=m'x'+n'y',$$

the origin being the same, then prove that

$$nn'(m^2+m'^2-1)=mm'(n^2+n'^2-1).$$

3. Use transformation of co-ordinates (two dimensional and rectangular) to establish the following reciprocal relation between algebra and geometry :—

(*a*) Every first degree equation represents a line.

(*b*) The equation of any line is of first degree.

4. Show that in the transformation of the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

from one set of rectangular axes to another, the quantities $(a+b)$ and $(ab-h^2)$ remain unaltered.

5. Show that if the lines $4x-3y+2=0$ and $3x+4y-1=0$ be taken as the axes of X and Y, the equation

$$16x^2-24xy+9y^2-104x-172y+44=0$$

transforms to $Y^2=8X$, the original axes being rectangular.

6. Prove that the value of g^2+f^2 in the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ remains unaffected by orthogonal transformation without change of origin.

Hint. The first degree terms in the transformed equation are $2(g\cos\theta+f\sin\theta)X+2(f\cos\theta-g\sin\theta)Y+c$. Square and add the coeffs. of X and Y.

7. Prove that the transformation of rectangular axes which converts $\frac{X^2}{p}+\frac{Y^2}{q}$ into $ax^2+2hxy+by^2$ will convert

$$\frac{X^2}{p-\lambda}+\frac{Y^2}{q-\lambda}$$

into
$$\frac{ax^2+2hxy+by^2-\lambda(ab-h^2)(x^2+y^2)}{1-(a+b)\lambda+(ab-h^2)\lambda^2}.$$

Hint. $\frac{1}{p}+\frac{1}{q}=a+b$; $\frac{1}{pq}=ab-h^2$.

Now, $$\frac{X^2}{p-\lambda}+\frac{Y^2}{q-\lambda}=\frac{qX^2+pY^2-\lambda(X^2+Y^2)}{pq-\lambda(p+q)+\lambda^2}.$$

Divide above and below by pq; Substitute for $\frac{X^2}{p}+\frac{Y^2}{q}$, $\frac{1}{pq}$, $\frac{1}{p}+\frac{1}{q}$ and get the result.

CHAPTER IV

PAIRS OF STRAIGHT LINES

4·1. Equation of a pair of straight lines. In chapter II we have seen that the general equation of first degree in x and y represents a straight line. Let us consider what locus is represented by the equation

$$(Ax+By+C)(A'x+B'y+C')=0 \qquad \dots \quad (1).$$

Equation (1) is obviously satisfied by the co-ordinates of all points which lie on one or other of the lines

$$Ax+By+C=0,$$

and $$A'x+B'y+C'=0,$$

and it accordingly represents the above *pair of straight lines.*

If the two factors on the left hand side of (1) are multiplied together, we get an equation of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

where the values of a, b, c, f, g, h are easily calculable in terms of A, B, C, A', B', C'. This is the *most general equation of the second degree* and will represent a pair of straight lines provided the expression $ax^2+2hxy+by^2+2gx+2fy+c$ can be expressed as the product of two linear factors in x and y.

4·11. Pair of lines through the Origin. The equation $ax^2+2hxy+by^2=0$, which is a *homogeneous equation of the second degree*, represents a pair of straight lines passing through the origin for *all* values of a, h, and b.

For, solving for x,

$$x=\frac{-h\pm\sqrt{h^2-ab}}{a}y,$$

and the two lines represented by the homogeneous equation of the second degree are

$$ax+(h+\sqrt{h^2-ab})y=0,$$

and $$ax+(h-\sqrt{h^2-ab})y=0.$$

The lines are real and distinct, coincident, or imaginary according as $h^2 > = \text{or} < ab$.

4·2. Homogeneous Equation of the nth. degree. We shall prove that the equation

$$Ay^n+By^{n-1}x+Cy^{n-2}x^2+\dots\dots+Kx^n=0 \quad \dots \quad (1)$$

in which the degree of each term is n and which is accordingly called a homogeneous equation of the nth degree, represents n straight lines each of which passes through the origin.

Dividing each term in (1) by x^n,

$$A\left(\frac{y}{x}\right)^n+B\left(\frac{y}{x}\right)^{n-1}+C\left(\frac{y}{x}\right)^{n-2}+\dots+K=0.$$

This is an equation of the nth degree in $\frac{y}{x}$ and will have n roots which may be denoted by $m_1, m_2 \dots m_n$.

It should then be possible to write the equation as

$$A\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)\dots\left(\frac{y}{x}-m_n\right)=0,$$

or $$A\,(y-m_1x)\,(y-m_2x)\dots\dots\dots(y-m_nx)=0.$$

Therefore (1) is the locus of points lying on one or other of the n straight lines

$$y-m_1x=0, \quad y-m_2x=0, \ldots\ldots, \; y-m_nx=0.$$

Each of the above lines obviously passes through the origin.

The student familiar with the theory of equations will easily see that these n lines are not necessarily real and distinct.

4·3. Angle between the lines $ax^2+2hxy+by^2=0$.

Let the equations to the two lines represented by

$$ax^2+2hxy+by^2=0 \qquad \ldots \quad (1)$$

be $\qquad y=m_1x$ and $y=m_2x$.

Then, $\qquad (y-m_1x)(y-m_2x)=0,$

or $\qquad m_1m_2x^2-(m_1+m_2)xy+y^2=0$

is the same as (1).

Comparing coefficients,

$$\frac{m_1m_2}{a}=\frac{-(m_1+m_2)}{2h}=\frac{1}{b}.$$

Hence, $\qquad m_1m_2=\frac{a}{b},$

and $\qquad m_1+m_2=-\frac{2h}{b}.$

If θ be the angle between the given straight lines,

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{\{(m_1+m_2)^2-4m_1m_2\}}}{1+m_1m_2}.$$

Substituting for m_1m_2 and m_1+m_2,

$$\tan\theta=\frac{\sqrt{(4h^2-4ab)}}{a+b}$$

i.e., $$\theta=\tan^{-1}\left\{\frac{2\sqrt{(h^2-ab)}}{a+b}\right\}.$$

If $a+b=0$, the lines are perpendicular, and if $h^2=ab$, the lines are coincident.

4·31. (Angle between the pair of lines $ax^2+2hxy+by^2=0$, the axes being inclined at an angle ω.

Let the equation to the lines become

$$a'x^2+2h'xy+b'y^2=0$$

when transformed to rectangular axes.

If θ be the angle between the lines,

$$\tan\theta=\frac{2\sqrt{h'^2-a'b'}}{a'+b'}.$$

By invariants,

$$\frac{h^2-ab}{\sin^2\omega}=\frac{h'^2-a'b'}{\sin^2\frac{\pi}{2}},$$

and $$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}},$$

i.e., $h'^2-a'b'=\dfrac{h^2-ab}{\sin^2\omega}$, and $a'+b'=\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$.

Hence, $$\tan\theta=\frac{2\sin\omega\sqrt{h^2-ab}}{a+b-2h\cos\omega}.$$

If the lines are perpendicular,

$$a+b-2h\cos\omega=0.$$

Ex. 1. Prove that for all values of m the angle between the lines $(a+2hm+bm^2)\,x^2+2\,[(b-a)m-(m^2-1)\,h]xy+(am^2-2hm+b)\,y^2=0$ is the same.

[*Math. Tripos.*]

Ex. 2. If the distance of a given point (p, q) from each of two straight lines through the origin is k, prove that the equation of the straight lines is

$$(py-qx)^2=k^2(x^2+y^2).$$

Ex. 3. Prove that the equation of the straight lines which pass through the origin and make an angle α with the straight line $x+y=0$ is

$$x^2+2xy\sec 2\alpha+y^2=0.$$

Ex. 4. Show that the two straight lines

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$$

make with the axis of x angles α, β such that

$$\tan\alpha-\tan\beta=2.$$

Ex. 5. Prove that $bx^2-2hxy+ay^2=0$ represents two straight lines at right angles respectively to the straight lines $ax^2+2hxy+by^2=0$.

Ex. 6. Find the condition that one of the two lines

$$ax^2+2hxy+by^2=0$$

may be perpendicular to one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

Ans. $4(ah'+hb')(ha'+bh')+(aa'-bb')^2=0.$

Ex. 7. Prove that the product of the perpendiculars from the point (x', y') on the lines $ax^2+2hxy+by^2=0$ is equal to

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

Ex. 8. Show that if one of the straight lines given by the equation

$$ax^2+2hxy+by^2=0$$

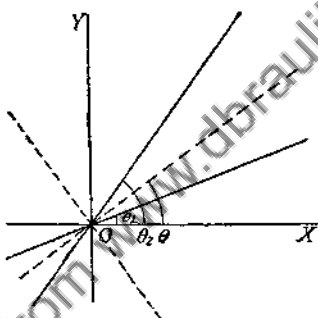
coincides with one of those given by

$$a'x^2+2h'xy+b'y^2=0,$$

then $$(ab'-a'b)^2+4\,(ah'-ha')\,(bh'-b'h)=0.$$

Hint. If $y=mx$ is the common straight line, then $a+2hm+bm^2=0$ and $a'+2h'm+b'm^2=0$. Eliminate m.

4·4. Equation to the bisectors of angles between the lines $ax^2+2hxy+by^2=0$.



Let θ_1 and θ_2 be the angles which the lines represented by the equation $ax^2+2hxy+by^2=0$ make with the axis of x. The slopes of the lines are then $\tan\theta_1$ and $\tan\theta_2$, and we have

$$\tan\theta_1+\tan\theta_2=-\frac{2h}{b},\ \tan\theta_1\tan\theta_2=\frac{a}{b}.$$

Let the dotted lines in the figure bisect the angles between the lines. If θ is the angle which one of the bisectors makes with the x-axis, then

$$\theta=\theta_1+\frac{\theta_2-\theta_1}{2}\ \text{ or }\ \frac{\pi}{2}+\theta_1+\frac{\theta_2-\theta_1}{2}$$

i.e., $$2\theta=\theta_1+\theta_2 \quad\text{or}\quad \pi+(\theta_1+\theta_2)$$

In either case, therefore, $\tan 2\theta=\tan(\theta_1+\theta_2)$.

Also, if (x, y) be the co-ordinates of any point on the bisector making an angle θ with x-axis,

$$\tan\theta=\frac{y}{x}.$$

$$\therefore \quad \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2xy}{x^2-y^2},$$

i. e.,

$$-\frac{2h}{b-a}=\frac{2xy}{x^2-y^2}.$$

Hence the equation to the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$ is

$$\mathbf{\frac{x^2-y^2}{a-b}=\frac{xy}{h}}$$

Ex. 1. Show that the bisectors of the angles between the lines

$$(Ax+By)^2=3(By-Ax)^2$$

are respectively parallel and perpendicular to the line $Ax+By+C=0$.

Ex. 2. Prove that the straight lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$$

have the same pair of bisectors. Interpret the case

$$\lambda=-(a+b).$$

Ex. 3. Show that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$$

is equal to the angle between the other two lines of the system.

Ex. 4. If pairs of straight lines

$$x^2-2pxy-y^2=0 \text{ and } x^2-2qxy-y^2=0$$

be such that each pair bisects the angles between the other pair, prove that $pq=-1$.

Ex. 5. Prove that the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0$$

is equally inclined to the pair of lines $ax^2+2hxy+by^2=0$.

Ex. 6. Show that the equation of the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$ is

$$\begin{vmatrix} \dfrac{ax+hy}{x+y\cos\omega} & \dfrac{hx+by}{y+x\cos\omega} \end{vmatrix}=0,$$

the axes being inclined at an angle ω.

4·5. Condition that the general equation of the second degree should represent a pair of straight lines.

Let the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \dots \quad (1)$$

represent a pair of straight lines and let (x_1, y_1) be the point of intersection of the lines.

If the origin is transferred to (x_1, y_1), the axes remaining parallel to their original directions, equation (1) transforms into

$$a(X+x_1)^2+2h(X+x_1)(Y+y_1)+b(Y+y_1)^2 +2g(X+x_1)+2f(Y+y_1)+c=0,$$

or $aX^2+2hXY+bY^2+2(ax_1+hy_1+g)X+2(hx_1+by_1+f)Y$

$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \dots \quad (2).$$

Since equation (2) represents a pair of straight lines passing through the origin for the new set of axes, it must be a homogeneous equation of the second degree in X and Y.

Hence the coefficients of X and Y and the constant term in (2) must separately vanish.

Thus,

$$ax_1+hy_1+g=0 \quad \ldots \quad (2),$$

$$hx_1+by_1+f=0 \quad \ldots \quad (3),$$

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \ldots \quad (4).$$

Multiplying (1) by x_1, (2) by y_1, adding, and subtracting the result from (3),

$$gx_1+fy_1+c=0 \quad \ldots \quad (5).$$

Eliminating x_1, y_1 from (2), (3) and (5),

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0,$$

which is the condition that equation (1) should represent a pair of straight lines.

Expanding the above determinant, the condition that the general equation of the second degree should represent two straight lines is

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0}$$

Corollary. The co-ordinates of the point of intersection of the straight lines represented by the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are obtained by solving for x_1 and y_1 the equations

$$ax_1+hy_1+g=0$$

$$hx_1+by_1+f=0.$$

Hence, $$x_1=\frac{hf-bg}{ab-h^2},\ y_1=\frac{gh-af}{ab-h^2}.$$

If $ab-h^2=0$, the point of intersection shifts to infinity and the lines become parallel.

The above method fails when the point of intersection of the lines lies at infinity. We may then proceed as follows :

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of straight lines if the expression on the left can be broken up into two linear factors of the type

$$lx+my+n \text{ and } l'x+m'y+n'.$$

We then have

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv(lx+my+n)(l'x+m'y+n'),$$

which gives

$$ll'=a,\ mm'=b,\ nn'=c,\ lm'+ml'=2h,$$
$$ln'+nl'=2g,\ mn'+nm'=2f.$$

Multiplying the last three results together,

$$2ll'mm'nn'+ll'(m^2n'^2+n^2m'^2)+mm'(n^2l'^2+l^2n'^2)$$
$$+nn'(l^2m'^2+m^2l'^2)=8fgh,$$

which reduces to

$$2abc+a(4f^2-2bc)+b(4g^2-2ac)+c(4h^2-2ab)=8fgh,$$

or
$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Ex. Prove that the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel straight lines if $h^2=ab$ and $bg^2=af^2$.

Prove also that the distance between them is

$$\frac{2\sqrt{(g^2-ac)}}{\sqrt{a(a+b)}}.$$

Substituting $h^2=ab$ in the general equation,

$$ax^2+2\sqrt{ab}xy+by^2+2gx+2fy+c=0,$$

or

$$(x\sqrt{a}+y\sqrt{b})^2+2(gx+fy)+c=0,$$

which will represent two straight lines obviously if

$$gx+fy\equiv\lambda(x\sqrt{a}+y\sqrt{b})$$

i. e., if

$$g=\lambda\sqrt{a},\quad f=\lambda\sqrt{b}.$$

i. e., if

$$\lambda=\frac{g}{\sqrt{a}}=\frac{f}{\sqrt{b}}$$

i. e., if

$$bg^2=af^2,$$

and the equation to the straight lines becomes

$$(x\sqrt{a}+y\sqrt{b})^2+\frac{2g}{\sqrt{a}}(x\sqrt{a}+y\sqrt{b})+c=0.$$

The two lines are therefore

$$x\sqrt{a}+y\sqrt{b}=\frac{-g\pm\sqrt{g^2-ac}}{\sqrt{a}}$$

or

$$x\sqrt{a}+y\sqrt{b}+\frac{g+\sqrt{g^2-ac}}{\sqrt{a}}=0 \quad \dots \quad (1)$$

and

$$x\sqrt{a}+y\sqrt{b}+\frac{g-\sqrt{g^2-ac}}{\sqrt{a}}=0 \quad \dots \quad (2)$$

The lines are obviously parallel. To determine the distance between them take any point (x, y) on one of them, say (2). The perpendicular from this point on (1) is

$$\frac{\sqrt{a}\,(x\sqrt{a}+y\sqrt{b})+g+\sqrt{g^2-ac}}{\sqrt{a}\,\sqrt{(a+b)}}$$

Since (x, y) lies on (2),

$$\sqrt{a}\,(x\sqrt{a}+y\sqrt{b})+g=\sqrt{g^2-ac}.$$

Hence the distance between the lines is

$$\frac{2\sqrt{(g^2-ac)}}{\sqrt{a(a+b)}}.$$

4·6. Sufficiency of the Condition. We have seen that if the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines,

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

so that $\Delta=0$ is the *necessary* condition that the general equation of the second degree should represent two straight lines. We shall show that this condition is *sufficient* also.

If $\Delta=0$, we can find x_1 and y_1 to satisfy the three linearly dependent equations

$$ax_1+hy_1+g=0$$

$$hx_1+by_1+f=0$$

$$gx_1+fy_1+c=0$$

If the origin is now transferred to (x_1, y_1) as determined by any two of the above equations, the general equation of the second degree is seen to reduce to

$$aX^2+2hXY+bY^2=0,$$

which is the equation to a pair of straight lines.

4·7. Angle between the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.

If the origin is transferred to the point of intersection of the lines, the equation reduces to

$$aX^2+2hXY+bY^2=0.$$

The lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are therefore parallel to the pair of lines

$$ax^2+2hxy+by^2=0$$

through the origin.

The angle between the lines is then

$$\tan^{-1}\left\{\frac{2\sqrt{h^2-ab}}{a+b}\right\}.$$

The lines are therefore *parallel* if $h^2=ab$, and *perpendicular* if $a+b=0$.

Alliter. If the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

be $\quad Ax+By+C=0, \quad$ and $\quad A'x+B'y+C'=0,$

we have $\quad (Ax+By+C)(A'x+B'y+C')\equiv k(ax^2+2hxy+by^2+2gx+2fy+c),$

so that $\quad (Ax+By)(A'x+B'y)\equiv k(ax^2+2hxy+by^2).$

Hence the lines represented by $ax^2+2hxy+by^2=0$ are $Ax+By=0$ and $A'x+B'y=0$ which are parallel to the lines represented by the general equation of the second degree.

The angle between the lines is therefore

$$\tan^{-1}\left\{\frac{2\sqrt{h^2-ab}}{a+b}\right\}.$$

Ex. 1. Show that the equation

$$3x^2+10xy+8y^2+14x+22y+15=0$$

represents two straight lines intersecting at an angle $\tan^{-1}\frac{2}{11}$.

Ex. 2. Find λ in order that the equation

$$x^2+\lambda xy+y^2-5x-7y+6=0$$

may represent two straight lines. Write down the equations to the lines parallel to these and passing through the point (1, 1).

Ans. $\frac{5}{2}, \frac{10}{3}$; $2x^2+5xy+2y^2-9x-9y+9=0$,

$3x^2+10xy+3y^2-16x-16y+16=0$.

Ex. 3. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of straight lines intersecting in (x_1, y_1), then the equation to the lines bisecting the angles between them will be

$$\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h}.$$

Ex. 4. Show that the four lines given by the equations

$2x^2+3xy-2y^2=0$ and $2x^2+3xy-2y^2+3x+y+1=0$

form a square.

4·8. Equation to the lines joining the origin to the points of intersection of a given straight line and a given curve of the second degree.

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \dots \quad (1)$$

represents a conic section of which the straight line pair is a special case.

Let the equation to the given line be

$$lx+my=1 \qquad \dots \quad (2).$$

(2) will intersect (1) in two points. The equation to the straight lines passing through the origin and the points of intersection of (1) and (2) will be a homogeneous equation of the second degree satisfied by the co-ordinates of common points of (1) and (2).

Let us therefore use (2) to make (1) homogeneous. We get,

$$ax^2+2hxy+by^2+2(gx+fy)(lx+my)+c(lx+my)^2=0 \qquad \dots \quad (3)$$

The general values of x and y which satisfy (2), reduce (3) to the equation (1) of the given curve. Hence the straight lines represented by (3) pass through the points of intersection of (1) and (2).

Corollary. The equation to the n straight lines joining the origin to the points of intersection of a curve of the nth degree and a straight line is obtained by making the equation of the curve homogeneous with the help of the equation of the straight line.

Ex. 1. Show that the straight lines joining the origin to the points of intersection of the line $x+2y-3=0$ and the circle $x^2+y^2-2x-2y=0$ are at right angles to one another.

Let us write $x+2y-3=0$ as $\dfrac{x+2y}{3}=1.$

Making the equation $x^2+y^2-2x-2y=0$ homogeneous with the help of this, we get

$$x^2+y^2-\frac{2(x+y)(x+2y)}{3}=0,$$

or
$$x^2-6xy-y^2=0.$$

Since the sum of the co-efficients of x^2 and y^2 is zero, the lines are at right angles.

Ex. 2. Prove that the angle between the straight lines joining the origin to the intersection of the straight line $y=3x+2$ with the curve $x^2+2xy+3y^2+4x+8y-11=0$ is $\tan^{-1}(\frac{2}{3}\sqrt{2})$.

Ex. 3. Obtain the equation to the pair of straight lines joining (h, k) to the points of intersection of the curve $ax^2+by^2=1$ with the line $axh+byk=1$.

Find the locus of (h, k) if the angle between the lines be $\dfrac{\pi}{2}$.

Ans. $(ax^2+by^2-1)(ah^2+bk^2-1)=(axh+byk-1)^2$;
$ab(x^2+y^2)=a+b.$

Ex. 4. Show that the straight lines joining the origin to the points of intersection of the two curves

$$ax^2+2hxy+by^2+2gx=0,$$

and

$$a'x^2+2h'xy+b'y^2+2g'x=0$$

will be at right angles to one another, if

$$g'(a+b)=g(a'+b').$$

Ex. 5. Show that all chords of the curve

$$3x^2-y^2-2x+4y=0$$

which subtend a right angle at the origin pass through a fixed point.

EXAMPLES ON CHAPTER IV

1. If a and b are positive numbers, for what range of values of k can a real λ be found such that the equation

$$ax^2+2\lambda xy+by^2+2k(x+y)+1=0$$

represents a pair of lines. [*Math. Tripos.*]

2. Show that the straight lines

$$(a^2-3b^2)x^2+8abxy+(b^2-3a^2)y^2=0$$

form with the line $ax+by+c=0$ an equilateral triangle of area

$$\frac{c^2}{(a^2+b^2)\sqrt{3}}.$$

3. The two equations

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

$$ax^2+2hxy+by^2-2gx-2fy+c=0$$

are each assumed to define a pair of straight lines.

Prove that these four lines enclose a parallelogram of area $\dfrac{2c}{\sqrt{h^2-ab}}$. [*Manchester.*]

4. If the lines

$$ax^2+2hxy+by^2=0$$

be two sides of a parallelogram and the line $lx+my=1$ be one of its diagonals, show that the equation of the other diagonal is

$$y\,(bl-hm)=x\,(am-hl).$$

5. A parallelogram is formed by the lines

$$ax^2+2hxy+by^2=0,$$

and the lines through (p, q) parallel to them. Prove that the equation of the diagonal which does not pass through the origin is

$$(2x-p)\,(ap+hq)+(2y-q)\,(hp+bq)=0.$$

Show also that the area of the parallelogram is

$$\frac{ap^2+2hpq+bq^2}{2\sqrt{h^2-ab}}.$$ [*U.P.C.S.*]

Hint. The straight lines parallel to

$$S_1\equiv ax^2+2hxy+by^2=0$$

through (p, q) are

$$S_2\equiv a\,(x-p)^2+2h(x-p)\,(y-q)+b(y-q)^2=0.$$

The required equation is $S_1-S_2=0$.

6. A variable line drawn through a fixed point (h, k) cuts the lines $px^2+2rxy+qy^2=0$ at A and B. Show that the middle point of AB lies on the curve

$$px^2+2rxy+qy^2=h(px+ry)+k(rx+qy).$$

7. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of straight lines, prove that the square of the distance of their point of intersection from the origin is

$$\frac{c\,(a+b)-f^2-g^2}{ab-h^2}.$$

Show also that the lines are equidistant from the origin if

$$f^4-g^4=c(bf^2-cg^2).$$

What will be the distance of the point of intersection from the origin if the lines are at right angles to one another?

8. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of lines, prove that the area of the triangle formed by their bisectors and the axis of x is

$$\frac{\sqrt{(a-b)^2+4h^2}}{2h}\cdot\frac{ca-g^2}{ab-h^2}.$$

Hint. The equation to the bisectors is

$$\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h},$$

where (x_1, y_1) is the pt. of intersection

If α, β be the abscissae of the points in which they meet the x-axis, α, β will be the roots of the equation

$$\frac{(x-x_1)^2-y_1^2}{a-b}+\frac{y_1(x-x_1)}{h}=0,$$

from which $(\alpha-x_1)+(\beta-x_1)=-\frac{y_1(a-b)}{h}$,

$$(\alpha-x_1)(\beta-x_1)=-y_1^2$$

$\therefore\quad \alpha-\beta=\frac{y_1\sqrt{(a-b)^2+4h^2}}{h}$. The area of the triangle is

$$\frac{y_1^2\sqrt{(a-b)^2+4h^2}}{2h}.$$

9. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, prove that the product of the lengths of the perpendiculars from the origin on these straight lines is

$$\frac{c}{\sqrt{\{(a-b)^2+4h^2\}}}.$$

10. If p_1, p_2 be the perpendiculars from (x, y) on the straight lines $ax^2+2hxy+by^2=0$, prove that

$$(p_1^2+p_2^2)\{(a-b)^2+4h^2\}=2(a-b)(ax^2-by^2) + 4h(a+b)xy+4h^2(x^2+y^2).$$

11. A point moves so that the distance between the feet of the perpendiculars from it on to the lines

$$ax^2+2hxy+by^2=0$$

is a constant $2k$. Show that the equation of its locus is

$$(x^2+y^2)(h^2-ab)=k^2\{(a-b)^2+4h^2\}.$$

12. A triangle has the lines $ax^2+2hxy+by^2=0$ for two of its sides and the point (p, q) for orthocentre. Prove that the third side is

$$(a+b)(px+qy)=ap^2-2hpq+bp^2$$

13. Find the area of the triangle, the equations of whose sides are

$$a_rx+b_ry+c_r=0, \quad r=1,\ 2,\ 3.$$

Deduce that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is

$$\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}.$$

14. Prove that the equation.

$$(a+2h+b)x^2-2(a-b)xy+(a-2h+b)y^2=0$$

denotes a pair of straight lines each inclined at an angle of 45° to one or the other of the lines given by

$$ax^2+2hxy+by^2=0.$$

15. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $ax^2+2hxy+by^2=0$. Prove that the locus of its vertex is

$$(a+b)(x^2+y^2)+2h(fy+gx)+(a-b)(fx-gy)=0.$$

16. If $u \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the equation of the third pair of straight lines passing through the points where these meet the axes is

$$cu+4\,(fg-ch\,)xy=0.$$

17. The orthocentre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is (x', y'). Show that $x'/l=y'/m=\dfrac{a+b}{am^2-2hlm+bl^2}$.

18. If two of the lines given by

$$ax^3+3bx^2y+3cxy^2+dy^3=0$$

are at right angles, show that

$$a^2+3ac+3bd+d^2=0.$$

Hint. If $y=mx$ be one of the lines given by the equation, $a+3bm+3cm^2+dm^3=0$. Find the condition that the product of two of the roots is equal to -1.

19. Show that two of the straight lines represented by the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0$$

will be at right angles if

$$(b+d)(ad+be)+(a-e)^2(a+c+e)=0.$$

CHAPTER V

THE GENERAL EQUATION OF THE SECOND DEGREE

5·1. The Conic Section. If a cone having a circular base (not necessarily right) is cut by a plane, the section is one of the five curves *viz.* a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. The shape of the section depends upon the position of the cutting plane, for example, the section by a plane through the vertex will be a pair of straight lines and the section by a plane parallel to the base will be a circle. If the plane neither passes through the vertex nor is parallel to the base the section will be one of the curves—parabola, ellipse or hyperbola.

The above five curves are for this reason called the conic sections. They all share what is called the **focus—directrix** property. That is, each one of them is the locus of a point which moves such that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. The fixed point is called the **focus,** the fixed line the **directrix,** and the constant ratio the **eccentricity** which is denoted by the letter 'e'. The conic* is an ellipse, parabola or hyperbola according as $e<1$ $=1$ or >1. The pair of straight lines and the circle are limiting cases of the parabola and the ellipse respectively. The parabola too is a limiting case of the ellipse.

5·2. Equation to a Conic Section. We shall show that every conic section is represented by an equation of the second degree.

We have seen in the preceding chapter that a pair of straight lines is represented by an equation of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

* Conic Section is briefly written as conic.

We shall see in the next chapter that the same equation represents a circle under a different set of conditions.

If the section be an ellipse, or parabola, or hyperbola let e be the eccentricity, (h, k) the co-ordinates of the focus and $Ax+By+C=0$ the equation of the directrix referred to rectangular axes in the plane of the section. The distances of any point (x, y) on the curve under consideration from the focus and the directrix are respectively

$$\sqrt{(x-h)^2+(y-k)^2} \quad \text{and} \quad \frac{Ax+By+C}{\sqrt{A^2+B^2}}.$$

Hence

$$(x-h)^2+(y-k)^2=e^2\frac{(Ax+By+C)^2}{A^2+B^2} \qquad \dots \quad (1),$$

which is an equation of the second degree of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \dots \quad (2),$$

and which represents the equation to a conic section.

If the axes were inclined at an angle ω, instead of (1) we would have

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega$$

$$=e^2\frac{(Ax+By+C)^2\sin^2\omega}{(A^2+B^2-2AB\cos\omega)},$$

which again is an equation of the second degree of the same form as (2).

We shall prove the converse proposition, namely, that an equation of the second degree always represents a conic section, in a later chapter after the standard equations to the circle, parabola, ellipse and the hyperbola have been discussed. We may state here for the benefit of the student that the standard equation to the *circle* is $x^2+y^2=a^2$, the standard equation to the *parabola* is

$y^2=4ax$, the standard equation to the *ellipse* is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the standard equation to the *hyperbola* is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where a and b have assigned values. The student may now realise that a suitable transformation of co-ordinate axes will reduce the general equation of the second degree to one or other of the standard forms.

We now proceed to obtain certain equations which are equally applicable to all conics and as easy to deduce for the general equations as for the standard ones.

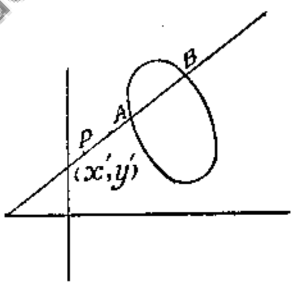
5·3. Intersection of a line and a conic. The equation to the line passing through the point P, (x', y'), is

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r \qquad \dots \quad (1)$$

where l and m are constants depending only on the direction of the line and r is the algebraical distance of P, *i.e.*, distance with proper sign, from (x, y).

The co-ordinates x and y of any point on the line are thus expressible as

$$x=x'+lr, \quad y=y'+mr.$$



To find where (1) meets the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

we substitute for x and y their values in terms of x', y', l, m and r. We then get

$$a(x'+lr)^2+2h(x'+lr)(y'+mr)+b(y'+mr)^2+2g(x'+lr)$$
$$+2f(y'+mr)+c=0,$$

or $$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l$$
$$+(hx'+by'+f)m\}+S'=0. \quad \dots \quad (2),$$

where $S'\equiv ax'^2+2hx'y'+by'^2+2gx'+2fy'+c.$

Equation (2) is of second degree in r and accordingly gives two values of r which may be real and distinct, real and coincident or imaginary depending upon the location of P with respect to the conic and the direction of the line.

Every straight line therefore cuts a conic in two points.

To the student familiar with Calculus equation (2) is rather simpler to deduce. For, if $S\equiv f(x, y)$,

$$f(x'+lr, y'+mr)=f(x', y')+r\left(l\frac{\partial f}{\partial x'}+m\frac{\partial f}{\partial y'}\right)$$
$$+\frac{r^2}{2}\left(\frac{\partial^2 f}{\partial x'^2}+2\frac{\partial^2 f}{\partial x'\partial y'}+\frac{\partial^2 f}{\partial y'^2}\right),$$

which immediately gives (2).

Imaginary points play as important a role in Analytical Geometry as the imaginary numbers do in other branches of Mathematics. To the student of pure geometry a line in the plane of a conic either meets it or does not meet it. In Analytical Geometry we are compelled to say that any line in the plane of a conic meets it in two points. This is so because we can have imaginary points, *i.e.*, points whose co-ordinates are of the form $p+iq$, p, q being real and $i=\sqrt{-1}$. Points with imaginary co-ordinates cannot be shown in a figure but their algebraical significance is a well established fact.

5·4. Tangent at any point (x′, y′) of the conic S=0.

We have seen above that a line cuts a conic in two points. If the two points of intersection coincide, the line touches the conic at either point and is called a tangent to the conic.

The distances PA and PB of the points of intersection A and B of the conic $S=0$ and the line

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r \qquad \dots (1)$$

from the point (x', y') are given by § 5.3

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0.$$

If the line is a tangent to the conic at (x', y') both the roots of this quadratic equation in r must be zero, which gives

(i) $S'=0$.

(ii) $(ax'+hy'+g)\,l+(hx'+by'+f)\,m=0$.

Relation (i) states that the point (x', y') lies on the conic $S=0$, and relation (ii) gives the ratio $l:m$ in order that the line may touch the conic.

From equation (1),

$$l:m=(x-x'):(y-y').$$

Relation (ii) then gives, on eliminating $\frac{l}{m}$,

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0,$$

This is the equation to the tangent at (x', y') and can be written as $(x-x')\frac{\partial f}{\partial x'}+(y-y')\frac{\partial f}{\partial y'}=0$

where $\qquad f(x, y)\equiv S.$

We shall reduce the equation to the tangent to what is known as the 'standard form'.

The equation can be written as

$$x\,(ax'+hy'+g)+y\,(hx'+by'+f)=x'\,(ax'+hy'+g)+y'\,(hx'+by'+f),$$

or

$$axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c$$
$$=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$$
$$=0,$$

which is the standard form, and is abbreviated into $T=0$.

The equation to the tangent at any point (x', y') of the conic $S=0$ is

$$\mathbf{T \equiv axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c=0.}$$

The following rule for writing the equation to the tangent at any point (x', y') of any conic may well be remembered.

(*i*) Replace x^2 by xx', y^2 by yy' without altering their co-efficients in the equation of the conic.

(*ii*) Replace $2xy$ by $xy'+yx'$.

(*iii*) Replace $2x$ and $2y$ by $x+x'$ and $y+y'$ respectively.

(*iv*) Retain the constant term.

5·41. Condition that the line lx+my+n=0 may be a tangent to the conic S=0. If the line touches the conic at (x', y'), the equation

$$lx+my+n=0$$

must be the same as

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0,$$

i.e., as $\quad x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c=0.$

Comparing co-efficients,

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}=\lambda, \text{ say}$$

Therefore,

$$ax'+hy'+g-l\lambda=0,$$
$$hx'+by'+f-m\lambda=0,$$
$$gx'+fy'+c-n\lambda=0.$$

We add to these

$$lx'+my'+n=0,$$

which is true since the point (x', y') lies on the given line.

Eliminating x', y' and λ,

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0,$$

which is the required condition. When expanded this gives $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$ where A, B, C etc. are the co-factors of a, b, c etc. in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The co-ordinates of the point of contact (x', y') are obtainable from the equations

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}.$$

Ex. 1. Show that the line $2x+2y-1=0$ touches the conic $x^2+y^2-xy+x+y-3=0$ at the point (1, 1.)

Ex. 2. Find the condition that the straight line $lx+my=1$ may touch the conic

$$(ax-by)^2-2(a^2+b^2)(ax+by)+(a^2+b^2)^2=0,$$

and show that if this straight line meets the co-ordinate axes in A and B, then AB will subtend a right angle at (a, b) when it is a tangent to the conic. *Ans.* $bl+am=0$.

5·5. Tangents that can be drawn from an external point (x′, y′) to the conic S=0. The co-ordinates of any point (x, y) on the line $\frac{x-x'}{l}=\frac{y-y'}{m}=r$ ... (1)

are given by $x=x'+lr,\ y=y'+mr.$

As before, the quadratic in r is

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0.$$

If the line is a tangent to the conic, the two values of r coincide, and we have,

$$S'(al^2+2hlm+bm^2)=\{(ax'+hy'+g)l+(hx'+by'+f)m\}^2.$$

Eliminating $l:m$ between this and (1),

$$S'\{a(x-x')^2+2h(x-x')(y-y')+b(y-y')^2\}$$
$$=\{(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')\}^2.$$

This can be written as

$$S'(S+S'-2T)=(T-S')^2,$$

which on simplification gives

$$\mathbf{SS'=T^2} \quad \dots (2)$$

which is a second degree equation in x and y.

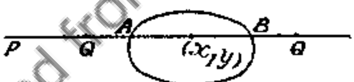
Equation (2) being the locus of points lying on a tangent to the conic, passing through (x', y'), shows that *from a point not lying on the conic there can always be drawn two tangents to the conic.* These tangents will be real or imaginary depending upon the co-efficients in the equation to the conic and the co-ordinates of the given external point. The points of contact of the tangents will be real or imaginary according as equation (2) represents a pair of real or imaginary lines.

The co-ordinates of the points of contact are obtained on solving $SS' = T^2$ and $S=0$ as simultaneous equations, *i.e.*, on solving the simultaneous equations $T=0$ and $S=0$.

If the point (x', y') lies on the conic itself the two tangents coincide, since then $S'=0$ and equation (2) reduces to $T=0$.

Alliter. The line $\dfrac{x-x'}{l} = \dfrac{y-y'}{m} = r$ ... (1)

through the point P, (x', y') cuts the conic in two points A and B. Let the co-ordinates of any other point on this line be (x, y).



If Q is a point on this line such that it divides the join of P and (x, y) internally or externally in the ratio $\lambda : 1$, the co-ordinates of Q will be

$$\left(\frac{\lambda x \pm x'}{\lambda \pm 1}, \frac{\lambda y \pm y'}{\lambda \pm 1} \right),$$

the plus sign being taken for internal division and the minus sign for external division.

If now Q lies on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

then $a\left(\frac{\lambda x \pm x'}{\lambda \pm 1}\right)^2 + 2h\left(\frac{\lambda x \pm x'}{\lambda \pm 1}\right)\left(\frac{\lambda y \pm y^1}{\lambda \pm 1}\right)$

$$+b\left(\frac{\lambda y \pm y'}{\gamma \pm 1}\right)^2 + 2g\left(\frac{\lambda x \pm x'}{\lambda \pm 1}\right) + 2f\left(\frac{\lambda y \pm y'}{\lambda \pm 1}\right)$$

$$+c = 0,$$

i.e., $a(\lambda x \pm x')^2 + 2h(\lambda x \pm x')(\lambda y \pm y') + b(\lambda y \pm y')^2$

$$+2g(\lambda x \pm x')(\lambda \pm 1) + 2f(\lambda y \pm y')(\lambda \pm 1) + c(\lambda \pm 1)^2 = 0.$$

Writing this in powers of λ,

$\lambda^2(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$

$\pm 2\lambda\{axx' + h(x'y + y'x) + byy' + g(x + x')$

$+f(y + y') + c\} + ax'^2 + 2hx'y' + by'^2 + 2gx'$

$+2fy' + c = 0,$

i. e., $\lambda^2 S \pm 2\lambda T + S' = 0.$

This is a quadratic equation in λ and shows that there will be two positions of Q which in the figure are shown by A and B. If the line (1) is a tangent to the conic the two values of λ will coincide for then there will be only one position of Q. We shall then have

$$SS' = T^2$$

which is therefore the equation to the pair of tangents from (x', y') to the conic $S = 0$.

The student will notice that this method gives the equation to the pair of tangents in the standard form without much calculation.

Ex. 1. Pairs of tangents are drawn to the conic $a'x^2 + b'y^2 = 1$ so as to be always parallel to the pair of straight lines $Ax^2 + 2Hxy + By^2 = 0$. If A, H and B vary in such a manner that $Ab-2Hh+Ba=0$, show that the locus of the point of intersection of these tangents is the conic

$$ax^2+2hxy+by^2=\frac{a}{a'}+\frac{b}{b'}.$$

The equation to the pair of tangents from (x', y') to the conic $a'x^2+b'y^2-1=0$ is

$$(a'x^2+b'y^2-1)(a'x'^2+b'y'^2-1)=(a'xx'+b'yy'-1)^2,$$

or
$$a'x^2(b'y'^2-1)-2a'b'x'y'xy+b'y^2(a'x'^2-1)$$
$$+\text{lower degree terms}=0.$$

Since these are parallel to the lines $Ax^2+2Hxy+By^2=0$,

$$\frac{A}{a'(b'y'^2-1)}=-\frac{H}{a'b'x'y'}=\frac{B}{b'(a'x'^2-1)}.$$

Eliminating A, H and B between these and the equation

$$Ab-2Hh+Ba=0,$$

$$a'b(b'y'^2-1)+2h\,a'b'x'y'+ab'(a'x'^2-1)=0$$

or
$$ax'^2+2hx'y'+by'^2=\frac{a}{a'}+\frac{b}{b'}.$$

Hence the required locus is

$$ax^2+2hxy+by^2=\frac{a}{a'}+\frac{b}{b'}.$$

Ex. 2. A pair of tangents to the conic $ax^2+by^2=1$ intercept a constant distance $2c$ on the axis of y; prove that the locus of their point of intersection is the curve

$$ax^2(ax^2+by^2-1)=bc^2(ax^2-1)^2.$$

Ex. 3. Find the equation of the pair of tangents drawn from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Show that the angle between them is

$$\tan^{-1}\left\{\frac{2ab\sqrt{\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1}}{x_1^2+y_1^2-a^2-b^2}\right\}.$$

Hence or otherwise find the locus of the vertices of equilateral triangles circumscribing the above ellipse.

Ans. $3(x^2+y^2-a^2-b^2)^2=4(b^2x^2+a^2y^2-a^2b^2)$.

Hint. The angle between the tangents will be $60°$.

5·6. Equation to the chord of contact of tangents that can be drawn from (x′, y′) to the conic S=0. We have seen that from any point (x', y') we can draw two tangents to a conic. The straight line joining the points of contact of these tangents is called the chord of contact.

Let the co-ordinates of the points of contact be (x_1, y_1) and (x_2, y_2).

The equation of the tangent at (x_1, y_1) is

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Since this passes through (x', y'), we have

$$ax'x_1+h(x'y_1+y'x_1)+by'y_1+g(x'+x_1)+f(y'+y_1)+c=0.$$

Similarly, since (x', y') also lies on the tangent at (x_2, y_2),

$$ax'x_2+h(x'y_2+y'x_2)+by'y_2+g(x'+x_2)+f(y'+y_2)+c=0.$$

From these two relations we easily conclude that (x_1, y_1) and (x_2, y_2) lie on the straight line

$$axx'+h(x'y+y'x)+byy'+g(x'+x)+f(y'+y)+c=0,$$

which is therefore the *equation to the chord of contact* of tangents that can be drawn from (x', y') to the conic

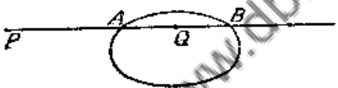
$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

This is the same as the equation to the tangent when (x', y') lies on the conic and can be written as $T=0$.

The chord of contact is a real line which by definition passes through the points of contact of tangents that can be drawn from the given point to the given conic. Since the points of contact may be imaginary as well, here we may have the case of a real straight line passing through two imaginary points.

5·7. Pole and Polar.

We now proceed to define the polar of a point with respect to a conic, and find its equation when the conic is represented by the general equation of the second degree.



Definition. If any secant, PAB, through a given point P, meets a conic in A and B, then the locus of Q, the harmonic conjugate of P with respect to A and B, *viz.*,

such that $\dfrac{1}{PA}+\dfrac{1}{PB}=\dfrac{2}{PQ}$ is the *polar* of P with respect to the conic and P is called its *pole*.

Let P and Q be the points (x', y') and (α, β) and let the equation to PAB be

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r.$$

As in § 5·3, r_1, r_2, the measures of PA and PB are the roots of

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0.$$

Let PQ be R. Then, since $\frac{1}{PA}+\frac{1}{PB}=\frac{2}{PQ}$,

$$R=\frac{2r_1 r_2}{r_1+r_2}=-\frac{S'}{(ax'+hy'+g)\,l+(hx'+by'+f)\,m}.$$

From the equation to the line,

$$\alpha-x'=lR,\quad \beta-y'=mR.$$

Therefore,

$$(ax'+hy'+g)\,(\alpha-x')+(hx'+by'+f)\,(\beta-y')+S'=0,$$

i.e., $a\alpha x'+h\,(\alpha y'+\beta x')+b\beta y'+g\,(\alpha+x')+f\,(\beta+y')+c=0.$

Hence the locus of R is the straight line

$$axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c=0$$

which is the *polar* of (x', y') with respect to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Corollary. If P is on the conic, the polar of P is the tangent at P.

The student will notice that the polar of a point is the chord of contact of tangents that can be drawn from the point to the conic. This is sometimes used as the definition of the polar. We shall however treat this only as a property of the polar.

The polar of a point with respect to a conic is also defined as the locus of the points of intersection of tangents at the extremities of chords through that point.

The student should find no difficulty in using this definition to obtain the equation to the polar of a point. For if (x', y') is the given point and (α, β) a point on the polar, (x', y') lies on the chord of contact of tangents from (α, β) to the conic, and therefore

$$a\alpha x'+h\,(\alpha y'+\beta x')+b\beta y'+g\,(\alpha+x')+f\,(\beta+y')+c=0$$

The locus of (α, β) as before is

$$axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c=0.$$

5·71. Conjugate Points. We shall show that *if the polar of P passes through Q then the polar of Q will pass through P.*

The polar of P, (x_1, y_1) with respect to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0.$$

If this passes through Q, (x_2, y_2),

$$ax_2x_1+h(x_2y_1+y_2x_1)+by_2y_1+g(x_2+x_1)+f(y_2+y_1)+c=0,$$

which is also seen to be the condition that P should lie on the polar of Q.

Two points such that each lies on the polar of the other are called conjugate points, and the condition that (x_1, y_1), (x_2, y_2) *should be conjugate points is*

$$ax_1x_2+h(x_1y_2+x_2y_1)+g(x_1+x_2)+f(y_1+y_2)+c=0.$$

5·72. Conjugate Lines. We shall prove that *if the pole of a line* $u_1=0$ *lies on another line* $u_2=0$, *then the pole of* $u_2=0$ *lies on* $u_1=0$.

Let (x_1, y_1), (x_2, y_2) be respectively the poles of $u_1=0$, $u_2=0$.

The line $u_1=0$ is the same as

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0 \ldots (1),$$

and $u_2=0$ is the same as

$$axx_2+h(xy_2+yx_2)+byy_2+g(x+x_2)+f(y+y_2)+c=0 \quad (2).$$

If, now, (x_1, y_1) lies on (2),

$$ax_1x_2+h(x_1y_2+y_1x_2)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0,$$

which is also the condition that (x_2, y_2) should lie on (1).

Hence if the pole of one line lies on another, the pole of the second line will lie on the first.

Two such lines are said to be *conjugate lines.*

5·73. The condition that two given lines should be conjugate with respect to a given conic.

Let
$$lx+my+n=0 \qquad \dots \quad (1),$$
$$l'x+m'y+n'=0 \qquad \dots \quad (2)$$
be conjugate lines for the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let (x', y') be the pole of (1) with respect to the conic, then (1) is the same as

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0,$$

i.e., as

$$x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c=0.$$

Comparing co-efficients,

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}=\lambda \text{ say}$$

Therefore,

$$ax'+hy'+g-l\lambda=0 \qquad \dots \quad (3),$$
$$hx'+by'+f-m\lambda=0 \qquad \dots \quad (4),$$
$$gx'+fy'+c-n\lambda=0 \qquad \dots \quad (5).$$

Also, (x', y') lies on (2) since (1) and (2) are conjugate lines.

Hence,
$$l'x'+m'y'+n'=0 \qquad \dots \quad (6).$$

Eliminating x', y' and λ from (3), (4), (5) and (6),

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l' & m' & n' & 0 \end{vmatrix} = 0,$$

which is the condition that the lines $lx+my+n=0$, and $l'x+m'y+n'=0$ should be conjugate with respect to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

5·74. Self-Polar Triangle. If the vertices A, B, C of a triangle are such that any pair of them are conjugate points, the triangle is said to be *self-polar* or *self-conjugate*. Any two sides of such a triangle will be conjugate lines. For since the polar of A passes through both B and C, BC is the polar of A, so that each side of the triangle is the polar of the opposite vertex. The rest is obvious as each vertex is the meeting point of two other sides of the triangle.

Ex. 1. Find the pole of the line $6x+y+5=0$ with respect to the conic $x^2+2xy-y^2+2x+4y-1=0$.

Ans. (2, 3).

Ex. 2. Show that the lines $lx+my+n=0$ and $l'x+m'y+n'=0$ are conjugate lines of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ if $a^2ll'+b^2mm'=nn'$.

Ex. 3. Show that the locus of poles with respect to the parabola $y^2=4ax$ of tangents to the hyperbola $x^2-y^2=a^2$ is the ellipse $4x^2+y^2=4a^2$.

5·8. The Centre. *The centre of a conic is a point such that all chords of the conic passing through it are bisected there.* We shall obtain the centre of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If (x', y') is the middle point of any chord whose equation is

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r,$$

then the chord will meet the conic in two points such that the algebraic sum of their distances from (x', y') is zero for all values of the ratio $l : m$.

As in § 5·3 the two values of r will be the roots of the equation

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0$$

Since the sum of the roots is zero, we have for all values of $l : m$

$$(ax'+hy'+g)l+(hx'+by'+f)m=0 \qquad \ldots (1)$$

Let $l_1 : m_1$ and $l_2 : m_2$ be two different values of $l : m$ for which (1) is true.

Then,

$$(ax'+hy'+g)\frac{l_1}{m_1}+hx'+by'+f=0,$$

and $$(ax'+hy'+g)\frac{l_2}{m_2}+hx'+by'+f=0.$$

Subtracting,

$$(ax'+hy'+g)\left(\frac{l_1}{m_1}-\frac{l_2}{m_2}\right)=0.$$

Since $$\frac{l_1}{m_1}\neq\frac{l_2}{m_2},$$

$$ax'+hy'+g=0,$$

and hence from (1),

$$hx'+by'+f=0.$$

The centre of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ *is therefore given by the equations*

$$ax+hy+g=0$$

$$hx+by+f=0,$$

and the co-ordinates of the centre are

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{gh-af}{ab-h^2}\right),$$

which can be written as

$$\left(\frac{G}{C},\ \frac{F}{C}\right)$$

where C, F and G are respectively the cofactors of c, f and g in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

If $ab-h^2=0$, the centre lies at infinity. The student will see in a later chapter that under this condition the conic is, in general, a parabola.

Alliter. The centre (x', y') of the general conic can also be found as follows :

If the origin is transferred to (x', y') the axes retaining their directions the equation to the conic becomes (§ 4·5)

$$aX^2+2hXY+bY^2+2(ax'+hy'+g)X+2(hx'+by'+f)Y+S'=0.$$

Since the points (X, Y), $(-X, -Y)$ now lie on the conic,

$$(ax'+hy'+g)X+(hx'+by'+f)Y=0.$$

And as this relation is true for all values of X and Y, we have as before the equations

$$ax'+hy'+g=0,$$

and $$hx'+by'+f=0,$$

which give the co-ordinates of the centre.

The student familiar with calculus will notice that the equations giving the centre are the ones obtained by differentiating the equation to the conic partially with respect to x and partially with respect to y.

5.81. The equation to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ when the origin is transferred to the centre.

The co-ordinates (x', y') of centre are given by

$$ax'+hy'+g=0 \qquad \ldots (1),$$

and $$hx'+by'+f=0 \qquad \ldots (2).$$

On transferring the origin to (x', y'), the equation of the conic becomes

$$aX^2+2hXY+bY^2+c'=0,$$

where $$c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c \qquad \ldots (3)$$

Multiplying (1) by x', (2) by y', adding and using (3),

$$gx'+fy'+c-c'=0 \qquad \ldots (4)$$

Eliminating x' and y' between (1) (2) and (4),

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-c' \end{vmatrix}=0.$$

or $$\Delta+\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & -c' \end{vmatrix}=0$$

or $$\triangle - c'(ab-h^2)=0,$$

i.e., $$c' = \frac{\triangle}{ab-h^2},$$

where $$\triangle \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The value of c' from (4) is $\mathbf{gx'+fy'+c}$, which is used when the co-ordinates of the centre are known.

Corollary. The equation $ax^2+2hxy+by^2=1$ represents a conic whose centre is at the origin.

Ex. Find the centre of the conic $x^2-5xy+y^2+8x-20y+15=0$ and determine its equation referred to parallel axes through the centre.

Ans. $(-4, 0)$, $x^2-5xy+y^2=1$.

5·9. Equation to the chord whose middle point is known. Let (x', y') be the middle point of a chord of the conic

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

The line $$\frac{x-x'}{l}=\frac{y-y'}{m}=r \qquad \ldots (1)$$

cuts the conic in two points whose distances from (x', y') are the roots of the equation

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0.$$

Hence, as in § 5·8,

$$(ax'+hy'+g)l+(hx'+by'+f)m=0$$

Eliminating $l : m$ between this and (1), *

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0,$$

which can be written as (§ 5·4)

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c$$
$$=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c.$$

The chord of the conic $S=0$ whose middle point is (x', y') is **T**=**S**′ where $T=0$ is the equation to the tangent at (x', y') when the point lies on the conic and S' is the value of S when x and y have been replaced by x' and y' respectively.

Ex. 1. Chords of the parabola $y^2=4ax$ subtend a right angle at the vertex (origin). Find the locus of their middle points.

If (x', y') is the middle point of one of the chords, its equation is

$$yy'-2a(x+x')=y'^2-4ax',$$

or

$$yy'-2ax=y'^2-2ax'$$

The equation to the pair of straight lines joining the origin to the points of intersection of the chord and the parabola is

$$y^2=4ax\ \frac{(yy'-2ax)}{y'^2-2ax'},$$

i.e.,

$$y^2(y'^2-2ax')+8a^2x^2-4ay'xy=0.$$

These are at right angles if

$$y'^2-2ax'+8a^2=0.$$

Hence the required locus is $y^2-2ax+8a^2=0$.

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*The elimination fails if $ax'+hy'+g=0$ and $hx'+by'+f=0$. In this case (x', y') is the centre of the conic and any chord through (x', y') satisfies the condition of the proposition.

Ex. 2 Find the locus of the middle points of chords of the conic $ax^2+by^2=1$ which subtend a right angle at the centre. *Ans.* $(a+b)(ax^2+by^2)^2=a^2x^2+b^2y^2$.

Hint. The centre of the conic is at the origin.

Ex. 3. Show that the tangent at an extremity of a diameter of a conic is parallel to the chords bisected by the diameter.

5·91. Locus of the middle points of a system of parallel chords of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Let a system of chords of the conic be parallel to the line $y=mx$. If (x', y') is the middle point of a chord of the system, the equation to the chord is

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=S',$$

i.e.,
$$x(ax'+hy'+g)+y(hx'+by'+f)+c=S'$$

This is parallel to $y=mx+c$, if

$$-\frac{ax'+hy'+g}{hx'+by'+f}=m.$$

Hence the required locus is the straight line

$$(ax+hy+g)+m(hx+by+f)=0.$$

This obviously passes through the point of intersection of the lines

$$ax+hy+g=0 \text{ and } hx+by+f=0,$$

which is the centre of the given conic.

5·92. Conjugate Diameters.

Definition. *The locus of the middle points of parallel chords of a conic is called a diameter. Two diameters are said to be conjugate when each bisects chords of the conic parallel to the other.*

We have seen in the preceding article that a diameter of a central conic passes through its centre. Let the diameter drawn parallel to $y=m'x$ bisect all chords of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, which are parallel to $y=mx$.

The equation to the diameter from the preceding article is

$$(ax+hy+g)+m(hx+by+f)=0,$$

or $$x(a+hm)+y(h+bm)+g+fm=0.$$

Since by hypothesis this is parallel to $y=m'x$, we have

$$-\frac{a+hm}{h+bm}=m'$$

i.e., $$a+hm+(h+bm)m'=0,$$

i.e., $$a+h(m+m')+bmm'=0.$$

From the symmetry of this relation in m and m' we see that the same is the condition in order that the diameter parallel to $y=mx$ may bisect chords parallel to $y=m'x$.

Hence *the diameters parallel to $y=mx$ and $y=m'x$ are conjugate if*

$$\mathbf{a+h(m+m')+bmm'=0.}$$

Ex. Find the condition that the lines $Ax^2+2Hxy+By^2=0$ may be conjugate diameters of the conic $ax^2+2hxy+by^2=1$.

Let $y=mx$ and $y=m'x$ be conjugate diameters of the given conic.

Then $(y-mx)(y-m'x)=0$ must be identical with $Ax^2+2Hxy+By^2=0$, which gives

$$m+m'=-\frac{2H}{B} \text{ and } mm'=\frac{A}{B}.$$

But, $$a+h(m+m')+bmm'=0.$$

Hence, $$a-h\frac{2H}{B}+b\frac{A}{B}=0$$

i.e., $$aB-2hH+bA=0,$$

which is the required condition.

EXAMPLES ON CHAPTER V

1. Show that the locus of the poles of the tangents to one conic A with respect to another conic B is a third conic.

2. Prove that the line joining two points in the plane of a conic is the polar of the point of intersection of their polars with respect to the conic.

3. Show that the locus of the points such that the chords of contact of tangents drawn from them to the conic $ax^2+2hxy+by^2=1$ subtend a right angle at the centre is the conic

$$x^2(a^2+h^2)+2h(a+b)xy+y^2(h^2+b^2)=a+b.$$

4. The two lines $x-\alpha=0$, $y-\beta=0$ are conjugate with respect to the hyperbola $xy=c^2$. Prove that (α, β) is on the hyperbola $xy-2c^2=0$.

5. The polar of the point P with respect to the parabola $y^2=4ax$ meets the curve in Q, R. Show that, if P lies on the straight line $Ax+By+C=0$, then the middle point of QR lies on the parabola

$$A(y^2-4ax)+2a(Ax+By+C)=0 \quad \text{[Math. Tripos 1911].}$$

6. Prove that any chord of a conic drawn through any point P will be cut harmonically by the conic and the polar of P.

7. Through the point P (α, β) two straight lines PLM, $PL'M'$ are drawn meeting the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

in the points L, M, L', M'. The lines LM', $L'M$ intersect in Q. Prove that the locus of Q is

$$(a\alpha+h\beta+g)x+(h\alpha+b\beta+f)y+g\alpha+f\beta+c=0,$$

when the directions of PLM, $PL'M'$ vary.

8. If the tangents drawn from external points to the conic represented by the general equation of the second degree be at right angles, show that the locus of their points of intersection is the circle

$$(ab-h^2)(x^2+y^2)-2(hf-bg)x-2(gh-af)y$$
$$+c(a+b)-f^2-g^2=0.$$

Show also that this circle is concentric with the conic.

Note. This is known as the *Director Circle.*

9. From a point P $(d\cos\theta, d\sin\theta)$, tangents PT, PT' are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Show that as θ varies, the middle point of the chord TT' describes the curve

$$x^2+y^2=d^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2.$$

[Manchester, 1945].

10. Tangents at right angles are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Show that the locus of the middle points of the chords of contact is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}.$$

11. Show that the locus of the middle points of chords of constant length $2c$ of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0.$$

12. Find the locus of the middle points of the chords of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

that are parallel to the line $\qquad lx+my+n=0.$

The ends of a chord are equidistant from a fixed point (x_0, y_0). Prove that the locus of the middle point of the chord is the conic

$$(x-x_0)(hx+by+f)-(y-y_0)(ax+hy+g)=0.$$

[I.C.S. 1941]

13. Prove that the co-ordinates of the centre of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are $\left(\frac{G}{C}, \frac{F}{C}\right)$, where $F\equiv gh-af$, $G\equiv fh-bg$, $C\equiv ab-h^2$.

If the diameter parallel to the tangent at P on the conic passes through the origin, prove that P lies on the diameter $g(Cx-G)+f(Cy-F)=0$.

14. Show that any two concentric conics have in general one and only one pair of common conjugate diameters.

Hint. Two concentric conics can be written as $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$. The diameters

$Ax^2+2Hxy+By^2=0$ are conjugate with respect to both if $Ab-2Hh+Ba=0$ and $Ab'-2Hh'+Ba'=0$, from which the equation to the common conjugate diameters is $(ha'-ah')x^2-(ab'-a'b)xy+(bh'-b'h)y^2=0$.

15. Show by transformation of axes that the equations of any two concentric conics can be written as

$$ax^2+by^2=1, \; a'x^2+b'y^2=1,$$

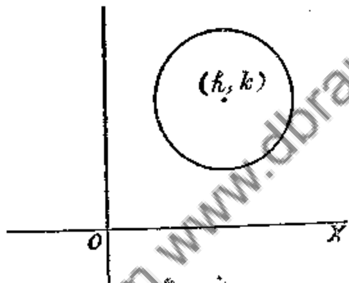
the axes of reference being oblique or rectangular.

CHAPTER VI

THE CIRCLE

6·1. Equation of a circle. Let (h, k) be the centre of a circle and a the radius. The equation to the circle is easily seen to be

$$(x-h)^2+(y-k)^2=a^2 \qquad \dots (1)$$



for it expresses the fact that the distance of any point (x, y) on the circle from the centre is a.

If the axes were oblique, instead of (1) the equation would have been $(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega = a^2$. We shall however work only in rectangular co-ordinates.

Transferring the origin to (h, k), equation (1) reduces to

$$x^2+y^2=a^2,$$

which represents a circle of which the centre is the origin and radius a. This form of the equation will be very frequently used.

Equation (1) can be written as

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0,$$

and since it represents any circle in the plane of rectangular axes we conclude that the general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a circle if $a=b$, and $h=0$.

The *general equation of a circle* is therefore

$$x^2+y^2+2gx+2fy+c=0,$$

where g, f, c are constants.

Writing this as

$$(x+g)^2+(y+f)^2=g^2+f^2-c,$$

and comparing with (1), the centre of this circle is $(-g, -f)$, and the radius $\sqrt{g^2+f^2-c}$.

The circle is real if $g^2+f^2>c$, imaginary if $g^2+f^2<c$, and a point circle if $g^2+f^2=c$.

6·11. Equation of the circle one of whose diameters is the line joining the points (x_1, y_1), (x_2, y_2). We shall use the property that the angle in a semi-circle is a right angle.

If (x, y) be any point on the circle, the slope of the line joining (x, y) and (x_1, y_1) is $\frac{y-y_1}{x-x_1}$, and the slope of the line joining (x, y) and (x_2, y_2) is $\frac{y-y_2}{x-x_2}$.

Since the two lines are at right angles,

$$\frac{(y-y_1)(y-y_2)}{(x-x_1)(x-x_2)}=-1,$$

or
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

which is the required equation.

6·12. The constants in the general equation of a circle. We have seen that the general equation of a circle is

$$x^2+y^2+2gx+2fy+c=0$$

If, therefore, three independent definite conditions are satisfied by the circle, the values of the three constants g, f and c are determined uniquely and the circle is also unique. For example, the equation $x^2+y^2-2px-2qy=0$ represents a circle passing through the origin. If, in addition, the circle touches the axis of x, $p=0$ for on putting $y=0$, the equation becomes $x^2-2px=0$ and the two roots of this are zero if $p=0$. The equation of a circle touching the axis of x at the origin is therefore $x^2+y^2-2qy=0$. A third condition is needed to determine q. If now we fix the radius of the circle q will be determined but not uniquely. But if the circle passes through a fixed point, the value of q will be unique.

Thus the circle touching the axis of x at the origin and having a radius equal to r is either $x^2+y^2-2ry=0$ or $x^2+y^2+2ry=0$. But the circle touching the axis of x at the origin and passing through (a, b) is only $b(x^2+y^2)-(a^2+b^2)y=0$.

A circle is determined uniquely if it passes through three given points not on the same straight line, but not uniquely if it touches three given straight lines. Thus the circle through the points $(0, 1)$, $(1, 0)$ and $(2, 1)$ is $x^2+y^2-2x-2y+1=0$ and the circles touching the straight lines $x+y=6$, $7x-y-42=0$, and $x-7y+42=0$ are $(2x-9)^2+(2y-9)^2=18$, $(x-2)^2+(y-12)^2=32$, $(x-12)^2+(y-2)^2=32$, and $(x+3)^2+(y+3)^2=72$.

The reader should verify these results by evaluating g, f and c in the two cases. In the second case he should use the property that the perpendicular from the centre upon a tangent is equal to the radius of the circle.

Let us now proceed to find the geometrical meaning of the constant c in the general equation of the circle, the meanings of g and f being already known.

Any line $\frac{x-x'}{l}=\frac{y-y'}{m}=r$ cuts the circle $x^2+y^2+2gx+2fy+c=0$ in points A and B whose distances from $P(x', y')$ are the roots of the equation

$$r^2(l^2+m^2)+2r\{(x'+g)l+(y'+f)m\}+x'^2+y'^2+2gx'+2fy'+c=0.$$

The product of the roots,

i.e., $$PA.PB=\frac{x'^2+y'^2+2gx'+2fy'+c}{l^2+m^2}$$

$$=x'^2+y'^2+2gx'+2fy'+c,$$

since $l=\cos\theta$, $m=\sin\theta$ where θ is the inclination of the line to the x-axis.

If $x'=0$, $y'=0$, $PA.PB=c$.

Thus the constant c is the rectangle under the segments of chords through the origin.

6·13. The point (x', y') and the circle $x^2+y^2+2gx+2fy+c=0$. The point (x', y') obviously lies on the circle $x^2+y^2+2gx+2fy+c=0$ if $x'^2+y'^2+2gx'+2fy'+c=0$. If it lies inside the circle, its distance from the centre is less than the radius; if it lies outside the circle, its distance from the centre is greater than the radius. Expressed analytically, the point (x', y') lies inside or outside the circle $x^2+y^2+2gx+2fy+c=0$, according as $(x'+g)^2+(y'+f)^2 <$ or $> g^2+f^2-c$,

i.e., according as $x'^2+y'^2+2gx'+2fy'+c<0$ or >0.

Thus $x^2+y^2+2gx+2fy+c<0$ *expresses analytically all points inside the circle* $x^2+y^2+2gx+2fy+c=0$, *and* $x^2+y^2+2gx+2fy+c>0$ *all points outside the circle.*

6·14. Equation of circle circumscribing a given triangle. Let the equations to the sides of the triangle be $a_rx+b_ry+c_r=0$, $r=1, 2, 3$.

The conic

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)+\lambda(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)$$
$$+\mu(a_3x+b_3y+c_3)(a_1x+b_1y+c_1)=0 \dots (1),$$

λ, μ being constants, passes through the vertices of the triangle for the co-ordinates of the points of intersection of any two of the given lines, which make two of the expressions $a_1x+b_1y+c_1$, $a_2x+b_2y+c_2$ and $a_3x+b_3y+c_3$ simultaneously zero, are easily seen to satisfy the equation to the conic.

Equation (1) can be written as

$$x^2(a_1a_2+\lambda a_2a_3+\mu a_3a_1)+xy\{(a_1b_2+a_2b_1)$$
$$+\lambda(a_2b_3+a_3b_2)+\mu(a_3b_1+b_3a_1)\}+y^2(b_1b_2+\lambda b_2b_3$$
$$+\mu b_3b_1)+\text{lower degree terms}=0.$$

This is a circle, if the co-efficients of x^2 and y^2 are equal and the co-efficient of xy is zero, which gives

$$a_1a_2-b_1b_2+\lambda(a_2a_3-b_2b_3)+\mu(a_3a_1-b_3b_1)=0 \dots (2),$$

and

$$a_1b_2+b_1a_2+\lambda(a_2b_3+a_3b_2)+\mu(a_3b_1+b_3a_1)=0 \dots (3).$$

Writing equation (1) as

$$\frac{1}{a_3x+b_3y+c_3}+\frac{\lambda}{a_1x+b_1y+c_1}+\frac{\mu}{a_2x+b_2y+c_2}=0,$$

and eliminating λ, μ from this, (2) and (3),

$$\begin{vmatrix} \dfrac{1}{a_3x+b_3y+c_3} & \dfrac{1}{a_1x+b_1y+c_1} & \dfrac{1}{a_2x+b_2y+c_2} \\ a_1a_2-b_1b_2 & a_2a_3-b_2b_3 & a_3a_1-b_3b_1 \\ a_1b_2+b_1a_2 & a_2b_3+b_2a_3 & a_3b_1+b_3a_1 \end{vmatrix}=0,$$

which is the equation of the required circle.

Ex. 1. Show that the locus of a point such that the ratio of its distances from two given points is constant is a circle.

Ex. 2. Whatever be the value of α, prove that the locus of the point of intersection of the straight lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$ is a circle.

Ex. 3. Show that the equation of the circle passing through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Ex. 4. Find the equation to the circle which passes through the points (5, 7) and (6, 6) and which has its centre on the straight line $2x-y-1=0$.

Ans. $x^2+y^2-4x-6y=12$.

Ex. 5. Lines drawn through the points (a, o), $(-a, o)$ make a constant angle α with one another. Show that the locus of their points of intersection are the circles

$$x^2+y^2-a^2 \pm 2ay \cot \alpha = 0.$$

Hence show that the angles in the same segment of a circle are equal.

Ex. 6. P is a point on the circle $x^2+y^2+2x+6y-15=0$, Q is a point on the line $7x+y+3=0$ and the perpendicular bisector of PQ is the line $x-y+1=0$, prove that there are two positions of the point P and find their co-ordinates.

[Math. Tripos 1941]

Ans. (3, 0), (−4, 1).

Ex. 7. Find the equation of the circle circumscribing the triangle formed by the lines

$$x+y=6,\ 2x+y=4,\ x+2y=5$$

The equation of any circle circumscribing the triangle formed by the given lines is

$$(x+y-6)(2x+y-4)+\lambda(2x+y-4)(x+2y-5)$$
$$+\mu(x+2y-5)(x+y-6)=0$$

where λ and μ are so chosen that the co-efficients of x^2 and y^2 are equal and the co-efficient of xy is zero.

Simplifying,

$$x^2(2+2\lambda+\mu)+xy(3+5\lambda+3\mu)+y^2(1+2\lambda+2\mu)$$
$$-x(16+14\lambda+11\mu)-y(10+13\lambda+17\mu)+24+20\lambda+30\mu=0$$

$\therefore \qquad 2+2\lambda+\mu=1+2\lambda+2\mu \qquad$ ...(1)

and $\qquad 3+5\lambda+3\mu=0 \qquad$ ...(2).

From (1), $\qquad \mu=1$

$\therefore$ from (2), $\qquad \lambda=-\frac{6}{5}$.

The equation to the circle therefore becomes

$$\tfrac{3}{5}x^2+\tfrac{3}{5}y^2-\tfrac{51}{5}x-\tfrac{57}{5}y+30=0$$

i. e., $\qquad x^2+y^2-17x-19y+50=0.$

6·2. Equation of tangent.

The equation of the tangent at a point (x', y') of the circle $x^2+y^2=a^2$ is (Chapter v)

$$xx'+yy'=a^2.$$

The line joining the centre to (x', y') is

$$xy'-yx'=0,$$

which is seen to be perpendicular to the tangent. This property may conveniently be used in the case of the circle to find the equation of the tangent.

Thus the tangent at (x', y') is the line which is perpendicular to the radius through (x', y').

The equation of the tangent at a point (x', y') of the circle

$$x^2+y^2+2gx+2fy+c=0$$

is
$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

6·21. Condition that the line $y=mx+c$ may be a tangent to the circle $x^2+y^2=a^2$.

The x co-ordinates of the points where the line cuts the circle are the roots of the equation

$$x^2+(mx+c)^2=a^2,$$

or
$$x^2(1+m^2)+2mcx+c^2-a^2=0.$$

The line touches the circle if the two points of intersection coincide *i.e.*, if the two values of x obtained from the above equation are coincident.

Hence,
$$m^2c^2+(a^2-c^2)(1+m^2)=0$$

i.e.,
$$c=\pm a\sqrt{1+m^2}$$

The line $y=mx+a\sqrt{1+m^2}$ is tangent to the circle $x^2+y^2=a^2$ for all values of m.

The value of c can be found out more easily by using the fact that the length of the perpendicular from the centre to the line $y=mx+c$ is equal to the radius of the circle. The student should find no difficulty in working out the details.

The co-ordinates of the point of contact are also easily obtained. If (x', y') is the point of contact,

$$xx'+yy'=a^2$$

and
$$y=mx+a\sqrt{1+m^2}$$

are the same.

Comparing co-efficients,

$$y' = -\frac{x'}{m} = -\frac{a}{\sqrt{1+m^2}}.$$

Hence the point of contact is

$$\left(\frac{-am}{\sqrt{1+m^2}}, -\frac{a}{\sqrt{1+m^2}} \right).$$

Ex. 1. Show that the line $x \cos \alpha + y \sin \alpha - a = 0$ touches the circle $x^2+y^2=a^2$.

Ex. 2. Find the tangents to the circle $x^2+y^2=a^2$ which are parallel to the line $\sqrt{3}x+y+3=0$.

Ans. $\sqrt{3}x+y\pm 2a=0$.

Ex. 3. Show that the line $x+y=1$ touches the circles $x^2+y^2+2x=1$ and $x^2+y^2+5x+3y-4=0$ at the same point.

Ex. 4. Find the condition that the straight line $lx+my+n=0$ should touch the circle

$$x^2+y^2+2gx+2fy+c=0.$$

Ans. $(c-f^2)\, l^2+2fglm+(c-g^2)m^2-2n\,(gl+fm)+n^2=0$.

Ex. 5. Find the equation to the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is

$$\alpha x+\beta y=0.$$

Ex. 6. Find the co-ordinates of the vertices of a rhombus, the sides of which are tangential to $x^2+y^2=2x$. Its longer side is $\sqrt{2}$ times the smaller one in length and is parallel to y—axis. Also show that the area of the rectangle obtained by joining the points of contact is $\frac{4}{3}\sqrt{2}$ sq. units.

Ans. $(1, \pm\sqrt{3})$, $(1\pm\sqrt{\frac{3}{2}}, 0)$.

Ex. 7. AP, BQ are parallel tangents to a circle, and a tangent at any point C cuts them in P and Q respectively. Show that $CP.CQ$ is independent of the position of the point C. [Math. Tripos]

Ex. 8. The straight line $px+qy=1$ makes with

$$ax^2+2pxy+by^2=c$$

a chord which subtends a right angle at the origin. Show that

$$c(p^2+q^2)=a+b,$$

and that the chord envelopes a circle whose radius is

$$\sqrt{\frac{c}{a+b}}.$$

The equation to the pair of lines joining the origin with the intersections of

$$px+qy=1$$

and

$$ax^2+2pxy+by^2=c$$

is

$$ax^2+2pxy+by^2=c(px+qy)^2;$$

coeff. of $x^2=a-cp^2$

coeff. of $y^2=b-cq^2$

Hence the required condition is

$$a-cp^2=b-cq^2$$

i.e.,

$$c(p^2+q^2)=a+b \quad \ldots \quad (1)$$

Now, the line $px+qy=1$ can be compared with the general equation $y=mx+a\sqrt{1+m^2}$ of the tangent to a circle.

Comparing coeffs., we get

$$-\frac{p}{q}=m, \ \frac{1}{q}=a\sqrt{1+m^2}$$

$$\therefore \qquad a=\frac{1}{q\sqrt{1+m^2}}=\frac{1}{\sqrt{p^2+q^2}}=\sqrt{\frac{c}{a+b}}$$

from (1) above.

6·22. Normal. The normal at any point of a curve is the straight line drawn perpendicular to the tangent at that point.

Since the tangent at (x', y') to the circle $x^2+y^2=a^2$ is $xx'+yy'=a^2$, the equation to the normal at (x', y') which is a line perpendicular to the tangent at this point is

$$x'(y-y')-y'(x-x')=0,$$

i.e.,

$$x'y-y'x=0.$$

Corollary. The equation to the normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x', y') is

$$y(x'+g)-x(y'+f)+fx'-gy'=0.$$

The student should note that every normal to a circle passes through its centre.

Ex. Find the locus of the feet of normals from (h, k) to the circles $x^2+y^2-2\lambda x=0$, where λ is variable.

Hint. The feet of the normals will be the points where the line joining (h, k) to the centre meets the circle. The locus of (h, k) is obtained on eliminating λ between the equations

$$y-k=\frac{k}{h-\lambda}(x-h) \text{ and } x^2+y^2-2\lambda x=0.$$

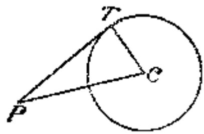
6·23. Length of the tangent from (x', y') to the circle $x^2+y^2+2gx+2fy+c=0$.

If P is the point (x', y'), C the point $(-g, -f)$, which is the centre of the given circle and PT a tangent through P, then the triangle PCT is right-angled at T, and

$$PT^2 = PC^2 - CT^2$$

$$= (x'+g)^2 + (y'+f)^2 - (g^2+f^2-c)$$

$$= x'^2 + y'^2 + 2gx' + 2fy' + c$$



Hence the required length of the tangent is

$$\sqrt{\mathbf{x'^2+y'^2+2gx'+2fy'+c}}$$

Corollary. The length of the tangent from (x', y') to the circle $ax^2+ay^2+2gx+2fy+c=0$ is

$$\sqrt{x'^2+y'^2+\frac{2g}{a}x'+\frac{2f}{a}y'+\frac{c}{a}}.$$

6·3. Pair of tangents. We have seen in the preceding chapter that we can draw two tangents from a given point to a conic. If the conic happens to be the circle $x^2+y^2=a^2$, the equation to the pair of tangents from (x', y') will be seen to be

$$(x^2+y^2-a^2)(x'^2+y'^2-a^2) = (xx'+yy'-a^2)^2.$$

The tangents touch the circle in points whose co-ordinates simultaneously satisfy the above equation and the equation

$$x^2+y^2-a^2=0.$$

Hence, the points of contact are given by

$$xx'+yy'-a^2=0$$

$$x^2+y^2-a^2=0$$

Eliminating y, $$x^2+\left(\frac{a^2-xx'}{y'}\right)^2-a^2=0,$$

i.e., $$x^2(x'^2+y'^2)-2a^2xx'+a^2(a^2-y'^2)=0.$$

The values of x will be real and distinct or imaginary according as

$$a^4x'^2 \begin{smallmatrix}>\\ \text{or}\\ <\end{smallmatrix} a^2(a^2-y'^2)(x'^2+y'^2),$$

i.e. $$0 \begin{smallmatrix}>\\ \text{or}\\ <\end{smallmatrix} y'^2(a^2-x'^2-y'^2).$$

Hence the tangents will be real and distinct if the point lies outside the circle and imaginary if the point lies inside the circle. If the point lies on the circle itself the tangents will be real and coincident.

The results of this article could also be obtained by finding the values of m for which the line $y=mx+a\sqrt{1+m^2}$ passes through (x', y'). It will be found that the values of m are real and distinct, real and coincident or imaginary according as the point lies outside, on or within the circle.

6·31. The cord of Contact. The chord of contact of tangents from (x', y') to the circle $x^2+y^2=a^2$ is (Chap. V)

$$xx'+yy'=a^2$$

If the point lies on the circle the chord of contact coincides with the tangent at the point. If the point is inside the circle the tangents from the point to the circle are imaginary, yet the chord of contact is a real line passing through the imaginary points of contact.

Ex. 1. Secants drawn from a given point P cut a given circle in point pairs A_1B_1; A_2B_2;......; A_nB_n.

Show analytically that $PA_1.PB_1 = \ldots\ldots = PA_n.PB_n = PT^2$ where PT is a tangent from P to the circle.

Ex. 2. Show that the equation to the pair of tangents drawn from the origin to the circle $x^2+y^2+2gx+2fy+c=0$

is $$(gx+fy)^2 = c\,(x^2+y^2).$$

Ex. 3. The distances from the origin of the centres of three circles $x^2+y^2-2\lambda x=c^2$ (where c is constant and λ a variable parameter) are in geometrical progression; prove that the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are also in geometrical progression.

Ex. 4. From any point on the circle

$$x^2+y^2+2gx+2fy+c=0$$

tangents are drawn to the circle

$$x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0.$$

Prove that the angle between them is 2α.

Ex. 5. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$. Prove that the area of the triangle formed by them and the straight line joining their points of contact is

$$\frac{a\,(h^2+k^2-a^2)^{3/2}}{h^2+k^2}.$$

The equation to the pair of tangents from (h, k) to the circle $x^2+y^2-a^2=0$ is

$$(x^2+y^2-a^2)\,(h^2+k^2-a^2)=(xh+yk-a^2)^2.$$

The angle α between these tangents is

$$\tan^{-1}\frac{2\sqrt{h^2k^2-(k^2-a^2)\,(h^2-a^2)}}{h^2+k^2-2a^2}$$

$$=\tan^{-1}\frac{2a\sqrt{h^2+k^2-a^2}}{h^2+k^2-2a^2}.$$

Therefore, $\sin\alpha = 2a\sqrt{h^2+k^2-a^2}/h^2+k^2$. The length of each tangent is $\sqrt{h^2+k^2-a^2}$.

Hence the area of the triangle is

$$\tfrac{1}{2}\sqrt{h^2+k^2-a^2}.\sqrt{h^2+k^2-a^2}.\frac{2a\sqrt{h^2+k^2-a^2}}{h^2+k^2}$$

$$=\frac{a\,(h^2+k^2-a^2)^{3/2}}{h^2+k^2}.$$

6·4. Pole and Polar. The polar of the point (x', y') with respect to the circle

circle $$x^2+y^2+2gx+2fy+c=0$$

is (Chap. V)

$$xx'+yy'+g\,(x+x')+f\,(y+y')+c=0 \quad \dots \quad (1)$$

The line joining the centre $(-g, -f)$ to (x', y') is

$$y+f=\frac{y'+f}{x'+g}\,(x+g) \quad \dots \quad (2)$$

Equation (1) can be written as

$$x\,(x'+g)+y\,(y'+f)+gx'+fy'+c=0.$$

The product of the slopes of this and (2) is equal to -1. Hence *the polar is perpendicular to the line joining the centre to the pole.*

The length of the perpendicular from the centre on (1) is

$$\frac{-c+g^2+f^2}{\sqrt{(x'+g)^2+(y'+f)^2}}.$$

Since the denominator is the distance of the pole from the centre, and the numerator the square of the radius of the circle, the pole and the point of intersection of the polar with the line joining the centre to the pole are *inverse* points with regard to the circle.

The inverse point of A with regard to a circle, centre C, is another point B if $CA.CB=(radius)^2$.

Again, if the polar of a point P passes through another point Q then the polar of Q will pass through P.

The student should prove this statement.

Two points such that each lies on the polar of the other are called conjugate points and two lines such that each contains the pole of the other are called conjugate lines.

There should be no difficulty in seeing that the points (x_1, y_1), (x_2, y_2) are conjugate with respect to the circle $x^2+y^2=a^2$ if $x_1x_2+y_1y_2=a^2$.

Ex. 1. Prove that the polar of a given point with respect to any one of the circles

$$x^2+y^2-2kx+c^2=0$$

where k is a variable always passes through a fixed point, whatever be the value of k.

If (x', y') is the given point, the polar is

$$xx'+yy'-k(x+x')+c^2=0,$$

which passes through

$$\left(-x', \frac{x'^2-c^2}{y'}\right).$$

Ex. 2. Find the locus of the poles of the line $lx+my+n=0$ with respect to circles which touch the y-axis at the origin.

The general equation of the circles touching the y-axis at the origin is

$$x^2+y^2+2hx=0,$$

If (α, β) be the pole, the polar is

$$x\alpha+y\beta+h(x+\alpha)=0,$$

or $$x(\alpha+h)+y\beta+h\alpha=0.$$

This must be the same as

$$lx+my+n=0.$$

Comparing coeffs.,

$$\frac{\alpha+h}{l}=\frac{\beta}{m}=\frac{h\alpha}{n},$$

i.e., $$h=\frac{n\beta}{\alpha m}.$$

Hence, $$\frac{\alpha+\frac{n\beta}{\alpha m}}{l}=\frac{\beta}{m},$$

or $$\alpha^2 m+n\beta=\alpha\beta l.$$

Hence the required locus is

$$mx^2+ny=xyl,$$

or $$x(ly-mx)=ny.$$

Ex. 3. Show that the lines $lx+my+n=0$ and $l'x+m'y+n'=0$ are conjugate with respect to the circle $x^2+y^2=a^2$

if $$(ll'+mm')\,a^2=nn'.$$

Ex. 4. Prove that the distances of two points from the centre of a circle are proportional to the perpendiculars drawn from one point on the polar of the other.

Ex. 5. *ABCD* is a quadrilateral inscribed in a circle. *AB*, *CD* meet in *O*; *AC*, *BD* in *P*, and *AD*, *BC*, in *Q*. Prove that *PQ* is the polar of *O*.

Ex. 6. Show that the locus of the poles of the line

$$\frac{x}{h}+\frac{y}{k}=1$$

with respect to the circles which touch the rectangular axes is given by the equations

$$(hx-ky)(hy-kx)+hk(h\pm k)(x\pm y)=0.$$

Ex. 7. The lengths of the tangents from two points A and B to a circle are l and l' respectively. If the points are conjugate with respect to the circle, show that

$$AB^2=l^2+l'^2.$$

6·5. Equation to the chord whose middle point is given. We shall use the property that the perpendicular from the centre upon any chord bisects it.

Let (x', y') be the middle point of a chord of the circle

$$x^2+y^2=a^2.$$

The equation to the straight line joining (x', y') to the centre is

$$xy'-yx'=0,$$

and the equation to a perpendicular to this through (x', y') is

$$y'(y-y')+x'(x-x')=0,$$

i.e.,

$$xx'+yy'=x'^2+y'^2$$

This therefore is the equation to the chord whose middle point is (x', y').

We could obtain this equation independently as in Chapter V for the general conic, and then we could deduce that the line joining the middle point of a chord of a circle to the centre is perpendicular to the chord.

Ex. 1. Find the middle point of the chord of the circle $x^2+y^2=a^2$ lying along the line $lx+my=n$.

If (α, β) is the middle point, the equation to the chord is

$$\alpha^2+\beta^2-a^2=x\alpha+y\beta-a^2$$

or

$$x\alpha+y\beta=\alpha^2+\beta^2.$$

This is the same as

$$lx+my=n.$$

Comparing coeffs.,

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\alpha^2+\beta^2}{n}=k \text{ say}$$

$\therefore$

$$\alpha=lk,\ \beta=mk.$$

And

$$\frac{(l^2+m^2)k^2}{n}=k$$

i.e.,

$$k=\frac{n}{l^2+m^2}.$$

Hence

$$\alpha=\frac{ln}{l^2+m^2},$$

$$\beta=\frac{mn}{l^2+m^2}.$$

Ex. 2. Show that the locus of the middle points of a series of parallel chords of a circle is a line through the centre.

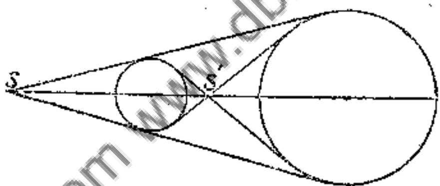
Ex. 3. Show that the locus of the middle points of the chords of contact of tangents drawn to a given circle from points on another given circle is a third circle.

Ex. 4. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which (*i*) pass through the fixed point (h, k), (*ii*) subtend a right angle at the point $(c, 0)$.

Ans. (*i*) $x^2+y^2-xh-yk=0$, (*ii*) $2(x^2+y^2)-2cx+c^2-a^2=0$.

6·6. Common tangents to two circles.

We know from pure geometry that we can draw two direct common tangents and two transverse common tangents to two circles which are such that one circle lies wholly outside the other. In the case of one circle touching the other the two tranverse common tangents coincide and we can draw three common tangents to the two circles. If the circles cut one another in real points, only the two direct common tangents can be drawn as real straight lines. In the limiting case of one circle lying wholly inside the other and touching at a real point, the direct common tangents coincide and only one real tangent can be drawn common to the two circles.



It will be seen that the two direct common tangents meet the line of centres in the same point S and the two transverse common tangents also meet the line of centres in the same point S', and that S and S' divide the line of centres externally and internally in the ratio of the radii of the circles.

The points S and S′ are called the Centres of Similitude.

Ex. 1. Find the common external tangents to the two circles $x^2+y^2=16$ and $x^2+y^2+6x-8y=0$.

The equation to the second circle can be written as

$$(x+3)^2+(y-4)^2=25$$

Its centre is therefore $(-3, 4)$, and radius 5.

The distance between the centres of the circles is 5, which is less than the sum of their radii which is 9 ; hence we can draw only the direct common tangents to the circles.

The point which divides externally the line joining (−3, 4) and (0, 0) in the ratio 5 : 4 is

$$\left(\frac{5.0+4.3}{5-4},\ \frac{5.0-4.4}{5-4}\right),\quad i.e., (12, -16).$$

The equation to any line through (12, −16) is

$$y+16=m(x-12).$$

This touches the circle $x^2+y^2=16$, if

$$\frac{12m+16}{\sqrt{1+m^2}}=\pm 4.$$

On solving,

$$m=\frac{-6\pm\sqrt{6}}{4}.$$

Hence the required tangents are

$$6x+4y-8=\pm\sqrt{6}(x-12).$$

Ex. 2. Find the equations of the common tangents of the circles

$$x^2+y^2-24x+2y+120=0$$

and

$$x^2+y^2+20x-6y-116=0$$

Ans. $24x+7y=156,\ 4x-3y=26$
$7y-24y=233,\ 3x+4y=57$

Ex. 3. Prove that the circles

$$x^2+y^2\pm 2ax+c=0$$

$$x^2+y^2\pm 2by+c=0$$

touch each other if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$.

Ex. 4. If $x \cos \theta + y \sin \theta = 2$ is the equation of the common tangent to $x^2+y^2=4$

and $x^2+y^2-6\sqrt{3}x-6y+20=0$,

find the value of θ. Prove also that the angle between the pair of direct common tangents to the two circles is $\tan^{-1}(4\sqrt{2/7})$. *Ans.* $\frac{1}{6}\pi$ or $\frac{1}{6}\pi \pm \cos^{-1}(-\frac{1}{3})$.

Ex. 5. Show that the common tangents of the circles

$$x^2+y^2+2x=0$$

and

$$x^2+y^2-6x=0$$

form an equilateral triangle.

6·7. Parametric Representation of the equation of a circle.

The co-ordinates of any point on the circle $x^2+y^2=a^2$ can be expressed as $x=a \cos \theta$, $y=a \sin \theta$, for on substitution $(a \cos \theta, a \sin \theta)$ satisfies the equation of the circle. Since now the circle can be regarded as the locus of a point which moves such that its abscissa is $a \cos \theta$ and ordinate $a \sin \theta$, where θ is a variable, we say that the equation to the circle in terms of the parameter 'θ' is

$$\mathbf{x}=\mathbf{a} \cos \theta$$
$$\mathbf{y}=\mathbf{a} \sin \theta$$

The point $(a \cos \theta, a \sin \theta)$ is, for brevity, called 'the point θ.'

It may be noted that θ is the angle which the line joining the point 'θ' to the centre of the circle makes with the x-axis.

Ex. 1. Find the equation to the straight line joining the points 'α' and 'β' on the circle $x^2+y^2=a^2$.

Hence deduce that the equation to the tangent at the point 'α' is $x \cos \alpha + y \sin \alpha = a$.

Ans. $x \cos \frac{\alpha+\beta}{2} + y \sin \frac{\alpha+\beta}{2} = a$.

Ex. 2. Show that the parametric equation

$$x=\frac{a(1-t^2)}{1+t^2}, \quad y=a\frac{2t}{1+t^2}$$

represents a circle of radius a.

EXAMPLES ON CHAPTER VI.

1. Show that the locus of a point which moves such that the sum of the squares of its distances from the vertices of a triangle is constant is a circle whose centre is the centroid of the triangle.

2. If the co-ordinates of the vertices of a quadrilatetal are $(0, \sqrt{3})$, $(1, 2\sqrt{3})$, $(4, \sqrt{3})$ and $(1, 0)$, prove that the quadrilateral is cyclic. Also find the co-ordinates of the centres and the radii of the inscribed and the circumscribing circles.

3. Show that the circle

$$x^2+y^2+4x-4y+4=0$$

touches the axes of co-ordinates.

Find the equations of the tangents which make equal intercepts on the axes of co-ordinates.

[Camb., 1939].

4. A variable chord PQ of a circle subtends a right angle at a fixed point, and C is the centre of the circle. Show that the feet of the perpendiculars from O and C to the line PQ lie on a circle whose centre is the middle point of OC.

5. Find the equation of a circle which touches the axis of y and passes through two points on the axis of x on the same side of the origin.

Two circles touch the axis of y and intersect in the points $(1, 0)$, $(2, -1)$. Find their radii and show that they both touch the line $y+2=0$. [Math. Tripos, 1912]

6. Find the equation of the circle circumscribing the triangle formed by the lines

$$ax^2+2hxy+by^2=0$$

and $$x\cos\alpha+y\sin\alpha-p=0.$$

Find also the area of the triangle and hence the product of the lengths of the sides of the triangle.

7. Show that the circle on the chord

$$x\cos\alpha+y\sin\alpha-p=0$$

of the circle $x^2+y^2-a^2=0$ as diameter is

$$x^2+y^2-a^2-2p\,(x\cos\alpha+y\sin\alpha-p)=0$$

Hint. The centre of the required circle is $(p\cos\alpha, p\sin\alpha)$ and the radius $\sqrt{a^2-p^2}$.

8. Show that the equation of a straight line meeting the circle $x^2+y^2=a^2$ in two points at equal distances d from a point (x_1, y_1) on the circumference is

$$xx_1+yy_1-a^2+\tfrac{1}{2}d^2=0.$$

Apply this method to find the equation of the tangent at (x_1, y_1).

9. From a point O on a fixed straight line are drawn two tangents to a given circle meeting in P, Q the tangent at A which is parallel to the tangent at either point where the fixed straight line meets the circle, prove that $AP+AQ$ is constant.

10. Circles are drawn through the point $(0, k)$ touching the circle $x^2+y^2=a^2$. Show that the locus of the pole of the axis of y with respect to these circles is the curve

$$4a^2\,(y-k)^4=(a^2-k^2)\,\{a^2-(k-2y)^2\}\,x^2.$$

11. Prove that the orthocentre of the triangle whose angular points are $(a\cos\alpha_r, a\sin\alpha_r)$, $r=1, 2, 3$, is the point $\left(a\sum_{r=1}^{3}\cos\alpha_r,\; a\sum_{r=1}^{3}\sin\alpha_r\right)$.

Hence prove that the centroid of any triangle divides the join of the circumcentre and orthocentre in the ratio 1 : 2.

12. If circles be described on the line joining the centres of similitude of two given circles as diameter, prove that the tangents drawn from any point on it to the two circles are in the ratio of the corresponding radii.

CHAPTER VII

Systems of Circles and Polar Equation.

7·1. Angle of Intersection. The angle of intersection of two curves at a point of intersection is defined to be the angle between the tangents to the curves at that point.

For example, the circles $x^2+y^2-2y=0$ and $x^2+y^2+2x-2y=0$ intersect at the origin at an angle of 45° since the tangents to them at the origin are respectively $y=0$ and $y=x$ which include an angle of 45°.

7·2. Orthogonal Circles. Two circles are said to cut orthogonally when the tangents at their points of intersection are at right angles.

We shall obtain the necessary and sufficient condition that the circles

$$x^2+y^2+2gx+2fy+c=0,$$

and

$$x^2+y^2+2g'x+2f'y+c'=0$$

should intersect orthogonally

Since the radii through a point of intersection are perpendicular to the tangents to the two circles at that point, the angle of intersection of the circles is the same as the angle between their radii through that point. If therefore the circles cut orthogonally the angle between the radii through a point of intersection will be 90° and then the square of the distance between their centres will be equal to the sum of the squares of their radii.

The centres of the given circles are $(-g, -f)$ and $(-g', -f')$ and their radii respectively $\sqrt{g^2+f^2-c}$ and $\sqrt{g'^2+f'^2-c'}$.

Hence, if the given circles cut orthogonally,

$$(g-g')^2+(f-f')^2=g^2+f^2-c+g'^2+f'^2-c',$$

i.e., $$\mathbf{2gg'+2ff'=c+c'}.$$

We thus see that this is the *necessary* condition for orthogonal intersection of two circles. Working the algebra backwards, it can be seen that this condition is *sufficient* also.

Ex. 1. Find the equation to the circle which passes through (1, 1) and cuts orthogonally each of the circles.

$$x^2+y^2-8x-2y+16=0 \text{ and } x^2+y^2-4x-4y-1=0.$$

Ans. $3x^2+3y^2-14x+23y-15=0$.

Ex. 2. Prove that two circles which pass through the points (o, a) and $(o, -a)$ and touch the line $y=mx+c$ will cut orthogonally if $c^2=a^2(2+m^2)$.

If (k, o) be the centre of one circle (the centre obviously lies on x-axis), the radius of the circle is $\sqrt{k^2+a^2}$.

The condition of tangency of $y=mx+c$ gives

$$\sqrt{k^2+a^2}=\pm\frac{mk+c}{\sqrt{1+m^2}}.$$

Squaring and simplifying,

$$k^2-2mck+a^2(1+m^2)-c^2=0 \qquad \dots \quad (1)$$

The circles cut orthogonally if the sum of the squares of their radii is equal to the square on their line of centres. Hence, if (k_1, k_2) be the roots of (1), the orthogonality condition is satisfied if

$$(k_1-k_2)^2=k_1^2+a^2+k_2^2+a^2,$$

i.e., if $$k_1k_2=-a^2.$$

Substituting for k_1k_2 from (1),

$$a^2(1+m^2)-c^2=-a^2$$

i.e., if $$(2+m^2)\,a^2=c^2.$$

Ex. 3. Show that if AB be a diameter of a circle, the polar of A with respect to the circle which cuts the first circle orthogonally will pass through B.

Ex. 4. If the equations of two circles whose radii are a, a' be respectively $S=0$, $S'=0$, then show that the circles

$$\frac{S}{a} \pm \frac{S'}{a'} = 0$$

will intersect at right angles.

7·3. Radical Axis. The radical axis of two circles is defined as the locus of points from where the lengths of the tangents drawn to the two circles are equal.

To find the radical axis of the circles

$$x^2+y^2+2gx+2fy+c=0,$$
$$x^2+y^2+2g'x+2f'y+c'=0,$$

we assume (x', y') to be any point from where the lengths of tangents drawn to these circles are equal.

Then

$$x'^2+y'^2+2gx'+2fy'+c=x'^2+y'^2+2g'x'+2f'y'+c',$$

i.e., $$2(g-g')x'+2(f-f')y'+c-c'=0.$$

Hence the locus of (x', y') is the straight line

$$\mathbf{2(g-g')x+2(f-f')y+c-c'=0},$$

which by definition is the *radical axis* of the given circles,

It may be noted that if $S=0$, $S_1=0$ are the equations of two circles such that the co-efficients of both x^2 and y^2 in each are the same say unity, the equation of the radical axis is $S-S_1=0$. But $S-S_1=0$ passes through the points of intersection of $S=0$, $S_1=0$. The radical axis of two circles is therefore the common chord of the circles. If the circles touch one another the radical axis is their common tangent at the point of contact.

7·31. Some Theorems on Radical Axis. We shall give below a few theorems connected with the radical axis.

Theorem I. *The radical axis of two circles is perpendicular to the line joining their centres.*

Let the two circles be

$$x^2+y^2+2gx+2fy+c=0$$

and
$$x^2+y^2+2g'x+2f'y+c'=0.$$

The centres of the circles are $(-g, -f)$, $(-g', -f')$.

The equation to the line of centres is

$$y+f=\frac{f-f'}{g-g'}(x+g).$$

Also, the radical axis is

$$2(g-g')x+2(f-f')y+c-c'=0.$$

The product of the slopes of these lines is

$$\frac{f-f'}{g-g'}\times\left(-\frac{g-g'}{f-f'}\right)=-1,$$

which shows that the radical axis is perpendicular to the line of centres.

Theorem II. *The radical axes of three circles, taken in pairs, meet in a point.* Let the equations of three circles be

$$S=0, S_1=0, S_2=0,$$

the co-efficients of x^2 and y^2 being unity in each equation.
The radical axes of the circles, taken in pairs, are

$$S-S_1=0,$$
$$S_1-S_2=0,$$
$$S_2-S_1=0.$$

Adding the left hand sides we find that their sum identically vanishes. The three lines represented by these equations are therefore concurrent.

The common point of the radical axes of three circles, taken in pairs, is called the *Radical Centre.*

Theorem III. *The difference of the squares of the tangents to two circles from any point in their plane varies as the distance of the point from their radical axis.*

If we take the line of centres of the circles as the axis of x, and the radical axis as the axis of y, the equations of the circles are written as

$$x^2+y^2+2gx+c=0,$$
$$x^2+y^2+2g'x+c=0.$$

For the radical axis is $2(g-g')x=0$, *i.e.*, $x=0$ and the centres are $(-g, 0)$, $(-g', 0)$, which evidently fulfils the conditions for this special choice of axes.

Now, let (x_1, y_1) be any point in the plane of the circles.

The difference of the squares of the tangents drawn from (x_1, y_1) to the circles is equal to

$$x_1^2+y_1^2+2gx_1+c-x_1^2-y_1^2-2g'x_1-c=2(g-g')x_1,$$

which varies as x_1, which is the distance of (x_1, y_1) from the axis of y, that is the radical axis.

Theorem VI. *If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.*

Chosing the line of centres of two circles as the axis of x, and their radical axis as the axis of y, the equations of the circles are

$$x^2+y^2+2gx+c=0 \ldots (1),$$

and

$$x^2+y^2+2g'x+c=0 \ldots (2).$$

Let $\quad x^2+y^2+2Gx+2Fy+C=0 \ldots (3)$

be a third circle which is cut orthogonally by (1) and (2).

We then have

$$2Gg+2F.0=c+C,$$

and $$2Gg'+2F.0=c+C.$$

From these, $G=0, \quad C=-c,$

and equation (3) reduces to

$$x^2+y^2+2Fy-c=0 \qquad \dots \quad (4).$$

The centre of (4) is $(0, -F)$, which lies on the axis of of y or the radical axis of (1) and (2).

Corollary. The radical centre of three circles is the centre of the circle which cuts them orthogonally, and the radius of this fourth circle is equal to the length of the tangent from the radical centre to any one of the three given circles.

Theorem V. *The radical axis of two circles bisects one of their common tangents.*

Theorem VI. *The radical centre of three circles described on the sides of a triangle as diameter is the orthocentre of the triangle.*

The proofs of the last two theorems are left as exercises to the student.

7. 4. The equations $S+\lambda S_1=0$, $S+\lambda u=0$.

If $S=0$ and $S_1=0$ be two circles, $S+\lambda S_1=0$, where λ is a constant, is also a circle for the co-efficients of x^2 and y^2 in $S+\lambda S_1=0$ are equal and the co-efficient of xy is zero. Further, the values of x and y which simultaneously satisfy $S=0$, $S_1=0$ obviously satisfy $S+\lambda S_1=0$.

Hence *$S+\lambda S_1=0$ is a circle which passes through the points of intersection of the circles $S=0$, $S_1=0$.*

If $S=0$, $S_1=0$ are two conics, $S+\lambda S_1=0$ is a conic which passes through the points of intersection of $S=0$, $S_1=0$.

Again, if $u=0$ is the equation to a straight line and $S=0$ the equation to a circle, $S+\lambda u=0$ is a circle since in this equation the co-efficients of x^2 and y^2 are equal, and there is no term in xy. This circle passes through the points of intersection of $S=0$ and $u=0$.

Corollary 1. If $S=0$ and $S_1=0$ be the equations of two circles, the equations of any two circles having the same radical axis as $S=0$ and $S_1=0$ are $S+\lambda_1 S_1=0$, $S+\lambda_2 S_2=0$.

Corollary 2. $S+\lambda u=0$ is the equation of a circle such that the radical axis of it and the circle $S=0$ is the line $u=0$.

Ex. 1. Find the radical centre of the three circles

$$x^2+y^2+4x+7=0,$$

$$2x^2+2y^2+3x+5y+9=0,$$

and

$$x^2+y^2+y=0.$$

The radical axes of the first and third circles and the second and third circles are respectively $4x-y+7=0$ and $x+y+3=0$. The lines meet in $(-2, 1)$, which is the radical centre of the given circles.

Ex. 2. Find the equation of the line joining the points of intersection of the circles represented by $x^2+y^2=4$ and

$$x^2-2x+y^2-4y+1=0,$$

and the length of the common chord.

[Cambridge, 1939.]

Since the radical axis is the common chord of two circles, the required equation is evidently

$$x^2+y^2-4-(x^2-2x+y^2-4y+1)=0,$$

i.e.,

$$2x+4y=5 \qquad \ldots \quad (1)$$

The length of the perpendicular from $(0, 0)$, the centre of one circle is $\frac{5}{\sqrt{20}}$. The radius of this circle being 2, the length of the common chord is

$$2\sqrt{4-\frac{25}{20}}$$

i.e. $\sqrt{11}$.

Ex. 3. Find the length of the common chord of the circles whose equations are $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$, and prove that the equation to the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay)$$

Ans. $\dfrac{2ab}{\sqrt{a^2+b^2}}$

Ex. 4. Find the general equation of the system of circles any pair of which have the same radical as the circles

$$x^2+y^2-2x-4y-4=0 \text{ and } x^2+y^2+8x-4y+6=0$$

Show that the equation to that member of the system which passes through the origin is $x^2+y^2+2x-4y=0$.

Ex. 5. Prove that by a proper choice of axes the equations of two circles can be written

$$x^2+y^2+2g_1x+c=0 \text{ and } x^2+y^2+2g_2x+c=0.$$

Show that the circles are orthogonal if $g_1g_2=c$ and that this can only happen for real circles if c is negative.

Ex. 6. Obtain the equation of the circle which cuts orthogonally the circle

$$x^2+y^2+6x+4y-3=0,$$

passes through (3, 0) and touches the axis of y.

Ans. $16x^2+16y^2-3x-24y+9=0.$

Ex. 7. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2-2x+3y-7=0,$$

$$x^2+y^2+5x-5y+9=0, \quad \text{and } x^2+y^2+7x-9y+29=0.$$

Ans. $x^2+y^2-16x-18y=4$.

Ex. 8. Show that the locus of points such that the difference of the squares of the tangents from them to two given circles is constant is a line parallel to their radical axis.

Ex. 9. If a circle cuts orthogonally three circles $S_1=0$, $S_2=0$ and $S_3=0$, prove that it cuts orthogonally any circle

$$\lambda S_1+\mu S_2+\nu S_3=0.$$

Hint. The condition of orthogonality

$$2g(\lambda g_1+\mu g_2+\nu g_3)+2f(\lambda f_1+\mu f_2+\nu f_3)$$
$$=(\lambda+\mu+\nu)c+(\lambda c_1+\mu c_2+\nu c_3)$$

is satisfied since the co-efficients of λ μ, ν vanish separately.

Ex. 10. Show that if $S_1=0$, $S_2=0$, $S_3=0$ be the equations of three circles of which each two cut orthogonally, the equation

$$l_1S_1+l_2S_2+l_3S_3=0$$

represents a real circle except in certain cases where it represents a straight line.

7.5. Coaxal Circles. Definition. *A system of circles in which the radical axis of each pair of circles is the same is called a Co-axal System.*

The centres of such a system of circles will all be in one straight line which will be perpendicular to the common radical axis because the line of centres of any pair of circles cuts the radical axis at right angles.

Choosing the line of centres to be the axis of x and the radical axis the axis of y, the equation to any member of the system has the form (Theorem III, § 7·31)

$$x^2+y^2+2gx+c=0 \qquad \dots \quad (1)$$

For different members of the system, g will be different but the constant c will have the same value.

If $g=\pm\sqrt{c}$, the radius g^2-c becomes zero, that is, the circles become point circles.

The point circles belonging to the coaxal system (1) are thus two circles of zero radius, with their centres at $(\pm\sqrt{c}, 0)$. The points $(\pm\sqrt{c}, 0)$ are known as the *limiting points* of the system under consideration.

Any two members of the system (1) are

$$x^2+y^2+2g_1x+c=0,$$

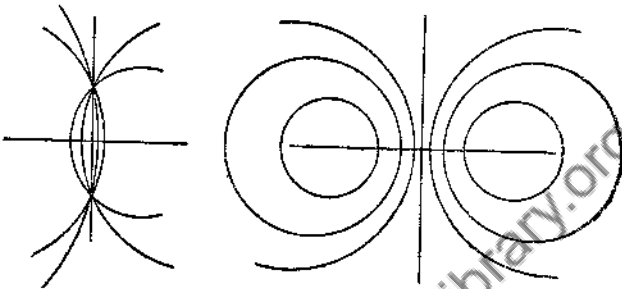
and $$x^2+y^2+2g_2x+c=0.$$

The x-co-ordinates of the points of intersection of these two circles are zero and the y co-ordinates are $\pm\sqrt{-c}$. If, therefore, c be positive no two members of the system intersect in real points although the limiting points are real. If c be the negative, the limiting points are imaginary, but any two members of the system intersect in real points which are the two points in which any member meets the radical axis of the system.

The limiting points are thus real or imaginary according as the circles of the system intersect in imaginary or real points. In the latter case the circles are said to be of the *Intersecting Species*.

Corollary 1. Circles passing through two fixed points form a coaxal system. For the common radical axis of any pair of such circles is the line joining the two given points.

Corollary 2. The equation to a coaxal system of which one member is the circle $S=0$ and of which the common



radical axis is the line $u=0$ is $S+\lambda u=0$ where λ is a constant. (See § 7.4)

Corollary 3. The equation to a coaxal system of which two members are $S=0$ and $S_1=0$ is $S+\lambda\, S_1=0$ where λ is a constant. (See § 7·4)

Ex. 1. Show that as λ varies the circles

$$x^2+y^2+ax+by+\lambda\,(lx+my+n)=0$$

form a coaxal system. Find the equation of the radical axis.

Ans. $lx+my+n=0$

Ex. 2. Find the limiting points of the coaxal system of which two members are the circles

$$x^2+y^2-6x-6y+4=0 \text{ and } x^2+y^2-2x-4y+3=0.$$

The equation of the coaxal system is

$$x^2+y^2-6x-6y+4+\lambda\,(x^2+y^2-2x-4y+3)=0,$$

or $$x^2+y^2-\frac{2\,(3+\lambda)\,x}{1+\lambda}-\frac{2\,(3+2\lambda)}{1+\lambda}\,y+\frac{4+3\lambda}{1+\lambda}=0.$$

The radius of this circle is

$$\frac{(3+\lambda)^2+(3+2\lambda)^2-(1+\lambda)\,(4+3\lambda)}{1+\lambda^2},$$

and the centre is

$$\left(\frac{3+\lambda}{1+\lambda}, \frac{3+2\lambda}{1+\lambda}\right)$$

The radius is zero, if

$$2\lambda^2+11\lambda+14=0,$$

The two values of λ which make the radius zero are therefore -2 and $-\frac{7}{2}$.

Hence the limiting points are $(-1, 1)$ and $(\frac{1}{5}, \frac{8}{5})$.

Ex. 3. Show that the limiting points of the coaxal system determined by the circles

$$x^2+y^2+2x+4y+7=0$$

and $$x^2+y^2+4x+2y+5=0$$

are $$(-2, -1) \text{ and } (0, -3),$$

Ex 4. Find the equation of the circle which passes through the origin and belongs to the coaxal system of which the limiting points are $(1, 2)$, $(4, 3)$.

Ans. $2x^2+2y^2-9x-13y=0.$

Ex. 5. Show that the locus of points, the tangents from which to two given circles bear a constant ratio is a coaxal circle.

Ex. 6. Show that the chords of intersection of a fixed circle with the circles of a given coaxal system pass through a fixed point. [Math. Tripos]

Ex. 7. Prove that the limiting points of a system of coaxal circles are inverse points with regard to every circle of the system.

7·51. The Polar of Limiting Points.

We shall show that the *polar of one limiting point of a coaxal system with respect to any circle of the system passes through the other limiting point.*

Taking the line of centres of the circles of the system as the x—axis and the radical axis as the y—axis, the equation to any circle of the system is

$$x^2+y^2+2gx+c=0\ldots(1).$$

The limiting points are $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$.

The polar of $(\sqrt{c}, 0)$ with respect to (1) is

$$x\sqrt{c}+g(x+\sqrt{c})+c=0,$$

that is $$(g+\sqrt{c})(x+\sqrt{c})=0.$$

Since $g\neq-\sqrt{c}$ the polar of $(\sqrt{c}, 0)$ is the line $x+\sqrt{c}=0$, which passes through the other limiting point $(-\sqrt{c}, 0)$.

We can, similarly show that the polar of $(-\sqrt{c}, 0)$ with respect to (1) passes through $(\sqrt{c}, 0)$.

The limiting points are therefore conjugate with respect to every member of the system.

7·52. The Orthogonal System.

We shall now obtain the equation of the circle which cuts orthogonally each member of a given system of coaxal circles.

Taking the axes as in the preceding article, the equation of any circle belonging to the given coaxal system is

$$x^2+y^2+2gx+c=0\ldots(1).$$

Let the circle whose equation is

$$x^2+y^2+2Gx+2Fy+C=0\ldots(2)$$

cut (1) orthogonally.

Then $$2gG=c+C.$$

If this relation is true for all values of g, G must be zero, and then $C=-c$.

Hence the equation of the circle which cuts every member of the given coaxal system orthogonally is

$$x^2+y^2+2Fy-\ =0\ldots(3).$$

Since F is arbitrary, we have a system of circles cutting the given system of coaxal circles orthogonally. The orthogonal system is also coaxal, the radical axis of any two circles of this system being the x—axis, and the line of centres the y—axis.

Every member of the orthogonal coaxal system is seen to pass through the points $(\pm\sqrt{c}, 0)$, the limiting points of of the given coaxal system.

The limiting points of the system represented by (3) are $(0, \pm\sqrt{-c})$ through which passes the system represented by (1). Further, if the limiting points of one system are real, those of the other system are imaginary.

The length of the tangent from $(0, -F)$, the centre of a member of the orthogonal system to (1) is $\sqrt{(F^2+c)}$, and is thus equal to the radius of the member of the orthogonal system.

Ex. 1. Show that the points which have the same polar with respect to every member of a system of coaxal circles are the limiting points of the system.

Ex. 2. A circle cuts orthogonally two fixed non-intersecting circles. Prove that it passes through two fixed points on their line of centres.

Prove also that the two points are inverse points with regard to either of the given circles. [Math. Tripos 1911]

Ex. 3. If the origin be at one of the limiting points of a system of coaxal circles of which

$$x^2+y^2+2gx+2fy+c=0,$$

is a member, the equation of the system of circles cutting them all orthogonally is

$$(x^2+y^2)(g+\mu f)+c(x+\mu y)=0.$$

The equation to the given coaxal system is

$$x^2+y^2+2gx+2fy+c+\lambda(x^2+y^2)=0.$$

Let
$$x^2+y^2+2Gx+2Fy+C=0$$

be the circle which cuts orthogonally every member of this system.

Then $$2Gg+2Ff=c+(1+\lambda)C$$

Since this is true for all values of λ, $C=0$, and

$$F=\frac{c-2Gg}{2f}.$$

Hence the equation to the orthogonal system is

$$x^2+y^2+2Gx+\frac{c-2Gg}{f}y=0.$$

or $$(x^2+y^2)+2G\left(x+\frac{c-2Gg}{2Gf}y\right)=0.$$

Writing $$\frac{c-2Gg}{2Gf}=\mu,$$

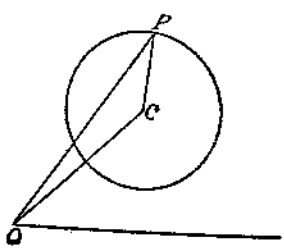
$$2G(g+\mu f)=c.$$

The required orthogonal system is, therefore,

$$(x^2+y^2)(g+\mu f)+c(x+\mu y)=0.$$

7·6. Polar Equation of a Circle.

Let the co-ordinates of the centre C of a circle be (c, α). Let a be the radius of the circle and P any point (r, θ) on the circle.



In the triangle OCP, $PC=a$, $OC=c$, $OP=r$ and the angle $POC=\theta-\alpha$.

Therefore, using the cosine formula,

$$\mathbf{a}^2=\mathbf{r}^2+\mathbf{c}^2-\mathbf{2cr}\cos(\theta-\alpha),$$

which is he equation of the given circle.

Corollary 1. The equation of the circle passing through the pole is $r=2a\cos(\theta-\alpha)$. For then $c=a$.

Corollary 2. The equation of the circle passing through the pole and having its centre on the initial line is $r=2a\cos\theta$. For in this case $c=a$ and $\alpha=0$.

Ex. 1. O is a fixed point and P any point on a given straight line ; OP is joined and on it is taken a point Q such that $OP.OQ=k^2$; prove that the locus of Q is a circle which passes through O.

Take O as the pole and a perpendicular from O to the given line as the initial line. The equation to the given line is then $p=r\cos\theta$.

Let (r_1, θ_1) be any point P on this line. The co-ordinates of any point Q on OP are (R, θ_1).

By the problem,

$$r_1 . R=k^2 \ldots (1).$$

But, since (r_1, θ_1) lies on $p=r\cos\theta$,

$$r_1=p\sec\theta_1$$

Substituting in (1), $R=\dfrac{k^2}{p}\cos\theta_1$

Hence the locus of Q is the circle

$$r=\frac{k^2}{p}\cos\theta.$$

The point Q is called the **inverse** of the given straight line with respect to O.

Ex. 2. Find the polar equation of the circle described on the straight line joining the points (a, α) and (b, β) as diameter.

Ans. $r^2 - r[a \cos(\theta-\alpha) + b \cos(\theta-\beta)] + ab \cos(\alpha-\beta) = 0.$

Ex. 3. Show that the straight line

$$\frac{1}{r} = A \cos\theta + B \sin\theta$$

touches the circle $r = 2a \cos\theta$, if $a^2B^2 + 2aA = 1$.

Ex. 4. Find the equation to the chord joining the points on the circle $r = 2a \cos\theta$ whose vectorial angles are α and β, and deduce the equation to the tangent at the point α. *Ans.* $r \cos(\theta-\alpha-\beta) = 2a \cos\alpha \cos\beta$.

Hint. Any line in polar co-ordinates is

$$p = r \cos(\theta-\theta').$$

This passes through the points $(2a \cos\alpha, \alpha)$ and $(2a \cos\beta, \beta)$, if $p = 2a \cos\alpha . \cos(\alpha-\theta') = 2a \cos\beta \cos(\beta-\theta')$ etc.

EXAMPLES ON CHAPTER VII

1. A certain point has the same polar with respect to each of two circles; prove that a common tangent subtends a right angle at this point.

2. Two circles intersect in the point A (x_1, y_1), and the line joining the other extremities of the two diameters through A makes an angle θ with the axis of x. Prove that the equation of the radical axis of the circles is

$$(x-x_1)\cos\theta + (y-y_1)\sin\theta = 0.$$

3. Two points P and Q are conjugate with respect to a given circle. Show that the circle on PQ as diameter is orthogonal to the given circle.

4. Find the co-ordinates of the limiting points of the coaxal system determined by the circles

$$x^2+y^2+2x-6y=0$$

and $$2x^2+2y^2-10y+5=0.$$

5. Prove that the polar lines of a fixed point P with respect to the circles of a given coaxal system pass through a fixed point Q. [Math. Tripos, 1942]

6. Given two circles, a tangent to one at P meets the polar of P with respect to the other in P'; prove that the circle on PP' as diameter will pass through two fixed points which will be imaginary or real as the given circles intersect in real or imaginary points.

7. If two circles cut orthogonally, prove that an indefinite number of pairs of points can be found on their common diameter, such that either point has the same polar with respect to one circle that the other has with respect to the other. Also show that the distance between any such pair of points subtends a right angle at one of the points of intersection of the two circles.

8. If the four points in which the two circles

$$x^2+y^2+ax+by+c=0,$$

$$x^2+y^2+a'x+b'y+c'=0$$

are intersected by the straight lines

$$Ax+By+C=0,$$

$$A'x+B'y+C'=0$$

respectively, lie on another circle, prove that

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0$$

Hint. The circles

$$x^2+y^2+ax+by+c+\lambda(Ax+By+C)=0,$$

and $$x^2+y^2+a'x+b'y+c'+\mu(A'x+B'y+C')=0$$

are identical.

9. Show that the general equation of all circles cutting at right angles the circles represented by

$$x^2+y^2-2a_1x-2b_1y+c_1=0,$$

$$x^2+y^2-2a_2x-2b_2y+c_2=0$$

is $$\begin{vmatrix} x^2+y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} + k \begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

10. If A, B, C be the centres of three coaxal circles, and t_1, t_2, t_3 be the tangents to them from any point, prove that

$$BCt_1^2+CAt_2^2+ABt_3^2=0.$$

11. If the circle

$$x^2+y^2-2\alpha x-2\beta y+c=0$$

is the circle of a coaxal system having the origin as a limiting point, show that the other limiting point is

$$\left(\frac{\alpha c}{\alpha^2+\beta^2}, \frac{\beta c}{\alpha^2+\beta^2}\right),$$

and prove that the equations of this system and the conjugate system are respectively

$$\lambda(x^2+y^2)-2\alpha x-2\beta y+c=0$$

$$(\alpha+\mu\beta)(x^2+y^2)-c(x+\mu y)=0,$$

λ, μ being variable parameters.

12. O is a fixed point. P is any point on a given circle. OP is joined and on it is taken a point Q such that $OP.OQ$ is a constant quantity. Prove that the locus of Q is a circle which degenerates into a straight line if the point O is taken on the original circle.

13. A circle passes through the point (r_1, θ_1) and touches the initial line at a distance c from the pole. Show that its polar equation is

$$\frac{r^2-2cr\cos\theta+c^2}{r\sin\theta}=\frac{r_1^2-2cr_1\cos\theta_1+c^2}{r_1\sin\theta_1}.$$

14. Find the centres of similitude of the circles

$$r^2-2ar\cos\theta+a^2\cos^2\alpha=0,$$

$$r^2-2br\cos\theta+b^2\cos^2\alpha=0$$

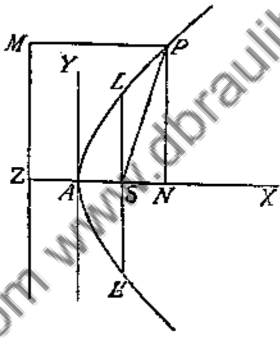
and show that the circle whose diameter is the line joining the centres of similitude is given by

$$(a+b)r=2ab\cos\theta.$$

CHAPTER VIII

THE PARABOLA

8·1. The Equation of parabola. *The parabola is the locus of a point which moves such that its distance from a fixed point (the focus) is equal to its distance from a fixed straight line (the directrix).*



Let S be the **focus** and ZM the directrix of the parabola.

Let SZ be the perpendicular from S on the directrix and A the middle point of SZ.

The point A lies on the parabola, since $AS=AZ$; and is called the **vertex** of the parabola.

Let AS and the perpendicular AY be chosen as the co-ordinate axes. If P be any point (x, y) on the parabola, $PS^2=(x-a)^2+y^2$, where $AS=a$. Also, PM the perpendicular on the directrix is equal to NZ where PN is the perpendicular on AS.

But $$NZ=AZ+AN=x+a.$$

Hence, $$(x-a)^2+y^2=(x+a)^2,$$

i.e., $$\mathbf{y^2=4ax}. \qquad \dots (1)$$

which is the standard equation of the parabola.

The line AS produced indefinitely is called the **axis** of the parabola. The perpendicular PN is called the ordinate of the point P, and the double ordinate LSL' through the focus is called the **latus rectum** of the parabola.

Since $AS=a$ and the point L lies on the parabola, from (1) $SL=2a$, $SL'=-2a$. The length of the latus rectum of the parabola represented by (1) is thus $4a$.

The equation to the *directrix* is $x+a=0$.

From equation (1) we see that corresponding to any positive value of x there are two equal and opposite values of y. That is, the points (x, y) and $(x, -y)$ both lie on the parabola.

The parabola is thus symmetrical with regard to its axis.

The parabola (1) does not extend on the negative side of x—axis for y is imaginary with negative x. Also, since x can have any positive value, as great as we choose, the parabola extends to infinity on the positive side of x—axis. If in (1) we put $x=0$, the two values of y are separately zero.

The y—axis is therefore tangent to parabola (1) at the vertex.

Further by a suitable transformation of axes, the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

can be reduced to the form (1) if it is the equation of a parabola.

By invariants, therefore,

$$ab-h^2=0,$$

which is the *condition* that the general equation of the second degree should represent a parabola.

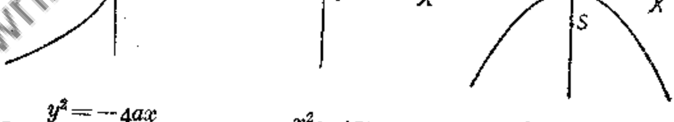
This condition is however not sufficient. *The general equation of the second degree may also represent a pair of parallel straight lines if* $ab-h^2=0$. *But if* $ab-h^2=0$

and
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0,$$

it represents only a parabola.

It can further be seen that the equation
$$y^2=-4ax \qquad \ldots (2)$$
represents an equal parabola lying wholly on the negative side of x-axis.

By rotating the axes through 90° we obtain from (1) and (2) the equations $x^2=-4ay$ and $x^2=4ay$ which accordingly represent equal parabolas of which the vertex is the origin and of which the axes lie respectively in the negative and positive directions of the y-axis.



$y^2=-4ax$
Latus rectum$=4a$

$x^2=4ay$
Latus rectum$=4a$

$x^2=-4ay$
Latus rectum$=4a$

It may be noted that the equation (1) expresses the fact that the square of the distance of any point on the parabola from the axis is equal to the product of the latus rectum and the distance of the point from the tangent at the vertex. This is a very important characteristic of the parabola.

Thus the equation

$$(ax+by+c)^2=k(bx-ay+c')$$

represents a parabola, whose axis is the line

$$ax+by+c=0,$$

and the tangent at the vertex is the line

$$bx-ay+c'=0.$$

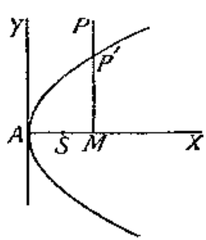
These two lines are at right angles.

Writing the equation to this parabola as

$$\left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2=\frac{k}{\sqrt{a^2+b^2}}\left(\frac{bx-ay+c'}{\sqrt{a^2+b^2}}\right),$$

we see that the latus rectum is the numerical value of the expression $\dfrac{k}{\sqrt{a^2+b^2}}$.

8·11. Position of a point with regard to the parabola $y^2=4ax$.



Let P be any point (x', y').

Draw PM perpendicular to x-axis and let it cut the parabola in P'.

The point P lies outside the parabola if $PM > P'M$.

But $PM = y'$, and $P'M^2 = 4ax'$.

Hence the point P lies outside the parabola if

$$y'^2 > 4ax'.$$

Similarly, the point (x', y') lies inside the parabola, if

$$y'^2 < 4ax'.$$

If, however, $y'^2 = 4ax'$, the point obviously lies on the parabola.

8·12. Parametric equation.

The co-ordinates of any point (x, y) on the parabola $y^2 = 4ax$ can be written as

$$\mathbf{x = at^2,\ y = 2\,at,}$$

where t is a *parameter*.

The point $(at^2, 2at)$ is for brevity, called the point 't'.

The parameter t will be seen to be the cotangent of the angle which the tangent at this point makes with the axis of x.

Ex. 1. Show that the focal distance of the point (x', y') on the parabola $y^2 = 4ax$ is $x' + a$.

Ex. 2. Trace the parabola

$$x^2 - 2ax + 4ay = 0.$$

The equation can be written as

$$(x - a)^2 = -4a(y - a).$$

Transferring the origin to (a, a), this becomes $x^2 = -4ay$, of which the vertex is $(0, 0)$ latus rectum $4a$, tangent at the vertex $y = 0$, and the axis $x = 0$ running along the negative direction of y-axis.

Referred to old axes, the vertex is (a, a), focus $(a, 0)$ tangent at the vertex $y = a$, and the axis $x = a$.

Ex. 3. Find the vertex, focus and directrix of the parabolas

$$(i)\ y^2+4x+4y-3=0\ ;$$

$$(ii)\quad x^2-2yb+b^2=a^2.$$

Ans. $(i)\ (\frac{7}{4},-2), (\frac{3}{4},-2), 4x=11\ ;$

$$(ii)\ \left(0, \frac{b^2-a^2}{2b}\right)\left(0, \frac{2b^2-a^2}{2b}\right), y=-\frac{a^2}{2b}.$$

Ex. 4. Find the equation of the parabola whose focus is $(-3, 0)$ and the directrix $x+5=0$. *Ans.* $y^2=4(x+4)$.

Ex. 5. An equilateral triangle is inscribed in the parabola $y^2=4ax$ so that one of its angular points is at the vertex. Find the sides of triangle. *Ans.* $8a\sqrt{3}$.

Ex. 6. A double ordinate of the parabola $y^2=4ax$ is of length $8a$; prove that the lines from the vertex to its two ends are at right angles.

Ex. 7. Prove that the chords of a parabola which subtend a right angle at the vertex meet the axis in a fixed point.

Let $y=mx+c$ be any chord of the parabola $y^2=4ax$.

The equation to the pair of lines joining origin with the intersections of the chord and the parabola is

$$y^2=\frac{4ax\ (y-mx)}{c}$$

i.e.,

$$cy^2-4axy+4amx^2=0$$

These lines are perp. if

$$c=-4am$$

The equation to the chord can, therefore, be written as

$$y=mx-4am,$$

or

$$y-m\ (x-4a)=0.$$

Since m is arbitrary, this passes through the intersection of $y=0$, $x-4a=0$. That is, $(4a, 0)$ is the fixed point through which the chord passes.

Ex 8. Show that the centre of the circle which touches the line $x+y=0$, and which passes through the point (a, a) lies on the parabola

$$x^2-2xy+y^2-4ax-4ay+4a^2=0.$$

Determine the co-ordinates of the vertex of this parabola.

Ans. $(a/2, a/2)$.

Ex. 9. A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; the axes being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is

$$\frac{x^2}{b^2}+\frac{y^2}{b^2-a^2}=1.$$

Ex. 10. A variable chord of the parabola $y^2=4ax$ passes through a fixed point. The circle on this chord as diameter cuts the parabola again at two other points. Prove that the line joining these two other points passes through another fixed point of which the ordinate is equal in magnitude to the ordinate of the first point.

8·2. Tangent and other Loci. As for the general conic in Chapter V, or regarding the parabola $y^2=4ax$ as a particular case of the general equation of the second degree we have the following results:

The tangent at a point (x', y') on the parabola is

$$yy'=2a(x+x').$$

The equation of the pair of tangents from (x', y') is

$$(y^2-4ax)(y'^2-4ax')=\{yy'-2a(x+x')\}^2.$$

The equation of the chord of contact of tangents from (x', y') or that of the polar of (x', y') is

$$yy' = 2a\,(x+x').$$

The equation of the chord whose middle point is (x', y') is

$$y'^2 - 4ax' = yy' - 2a\,(x+x'),$$

i.e.,

$$y'^2 - 2ax' = yy' - 2ax$$

8·3. Intersections of a straight line and a Parabola. The x—co-ordinates of the points of intersection of the line $y = mx + c$ and the parabola $y^2 = 4ax$ are the roots of the equation

$$(mx+c)^2 = 4ax,$$

or

$$m^2x^2 + 2x\,(mc - 2a) + c^2 = 0.$$

Being a quadratic in x, this will have two roots, which may be real and distinct, real and coincident, or imaginary. If $m \to 0$, one of the values of x tends to infinity.

We thus see that *any straight line cuts a parabola in two points. If the line is parallel to the axis of the parabola, one of the points of intersection is at infinity.*

If the roots of the above quadratic are real and coincident,

$$(mc - 2a)^2 = m^2c^2,$$

i.e.,

$$c = \frac{a}{m}.$$

The line $\mathbf{y = mx + \frac{a}{m}}$ *therefore is a tangent to the parabola* $y^2 = 4ax$ *for all values of* m. To find its point of contact, we compare it with the equation of the tangent at (x', y'), *viz.*, with the equation $yy' = 2a\,(x+x')$.

We then have

$$y' = \frac{2a}{m} = 2mx',$$

which gives $$x' = \frac{a}{m^2},\ y' = \frac{2a}{m}.$$

Hence the co-ordinates of the point of contact are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

Further, if the line $y = mx + \dfrac{a}{m}$ passes through a given point (h, k),

$$k = mh + \frac{a}{m},$$

or $$m^2h - mk + a = 0.$$

Since this is a quadratic in m, we can draw two tangents from a point (h, k) to the parabola $y^2 = 4ax$. These tangents will be real and distinct only if $k^2 > 4ah$, *i.e.*, if the point (h, k) lies outside the parabola.

Ex. 1. Find the equation to the chord of the parabola $y^2 = 4ax$ joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$. Hence deduce the equation to the tangent at the point 't'.

The line joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}\,(x - at_1^2),$$

or $$\tfrac{1}{2}(t_1 + t_2)\,y = x + at_1t_2.$$

If $t_2 \rightarrow t_1$, this becomes $t_1y = x + at_1^2$.

The tangent at the point 't' is thus $ty = x + at^2$.

Ex. 2. Tangents to the parabola $y^2=4ax$ include a constant angle α. Show that the locus of their points of intersection is the curve

$$y^2-4ax=(a+x)^2 \tan^2\alpha.$$

What happens to this curve if $\alpha=\frac{\pi}{2}$?

Ex. 3. Find the locus of the foot of the perpendicular drawn from a fixed point to any tangent to a parabola.

Ans. $x(x-h)^2+y(x-h)(y-k)+a(y-k)^2=0.$

Ex. 4. Two tangents OP and OQ are drawn to a parabola represented in rectangular cartesian co-ordinates by $y^2=4x$, from the point O, the co-ordinates of which are (4, 5). Prove that the line joining the midpoints of OP and OQ touches the parabola. [Cambridge, 1939]

Ex. 5. Show that $x-2y+4a=0$ is a tangent to the parabola $y^2=4ax$. What are the co-ordinates of the point of contact ?

The inclinations θ and φ of two tangents to the parabola $y^2=4ax$ are given by

$$\tan\theta=\frac{1}{m},\ \tan\varphi=\frac{m}{2}.$$

Show that as m varies, the point of intersection of these pairs of tangents traces a line parallel to the directrix.

[*Ans.* $4a$, $4a$].

Ex. 6. Show that the locus of the poles of chords of the parabola $y^2=4ax$ which subtend a constant angle θ at the vertex is the curve

$$4\,(y^2-4ax)=\tan^2\theta\ (x+4a)^2.$$

Ex. 7. Any chord of the parabola and the perpendicular to the chord through the pole of the chord meet the axis of the parabola at P and Q. Prove that P and Q are equidistant from the focus of the parabola.

[Birmingham, 1944]

Ex. 8. Prove that the area of the triangle formed by the tangents from the point (x_1, y_1) to the parabola $y^2=4ax$ and the chord of contact is

$$\frac{(y_1^2-4ax_1)^{3/2}}{2a}.$$

If the tangents at $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ meet at (x_1, y_1), $x_1=at_1t_2$ and $y_1=a(t_1+t_2)$. The length of the chord of contact is

$$a(t_1-t_2)\sqrt{(t_1+t_2)^2+4}$$

i.e.,
$$\frac{\sqrt{y_1^2-4ax_1}\ \sqrt{y_1^2+4a^2}}{a}.$$

The equation of the chord is

$$yy_1-2a(x+x_1)=0.$$

The perpendicular from (x_1, y_1) on the chord is

$$\frac{y_1^2-4ax_1}{\sqrt{y_1^2+4a^2}}.$$

Hence the area of the triangle required is

$$\frac{(y_1^2-4ax_1)^{3/2}}{2a}.$$

Ex. 9. Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Ex. 10. Chords of a parabola pass through a fixed point. Prove that the locus of their middle points is a parabola having its axis parallel to that of the given parabola. [Indian Audit & Accts. Service, 1942]

If (x', y') is the middle point of a chord of the parabola $y^2=4ax$, its equation is

$$yy'-2ax=y'^2-2ax'.$$

It passes through (h, k), a fixed point. Therefore,

$$ky'-2ah=y'^2-2ax',$$

Hence the locus of (x', y') is the parabola

$$y^2 - ky = 2a(x-h).$$

The axis of this parabola will be seen to be $y = \frac{k}{2}$.

8·4. The Normal.

The equation of the tangent at a point (x', y') of the parabola

$$y^2 = 4ax$$

is

$$yy' = 2a(x+x').$$

The equation of the normal at (x', y'), *i.e.*, the perpendicular to the tangent at (x', y') is

$$2a(y-y') + y'(x-x') = 0,$$

or

$$y - y' = -\frac{y'}{2a}(x-x').$$

Put $-\frac{y'}{2a} = m$, *i.e.*, $y' = -2am$.

Since $y'^2 = 4ax'$, $x' = am^2$.

Hence the *normal* at the point $(am^2, -2am)$ is

$$y + 2am = m(x - am^2),$$

i.e.,

$$\mathbf{y = mx - 2am - am^3}.$$

Since m is a parameter, the above equation represents any normal to the parabola $y^2 = 4ax$.

Instead of using the above form of the equation of the normal we could obtain the normal at the point $(at^2, 2at)$, which is

$$y + tx = 2at + at^3.$$

We can use *any* of these two forms in problems on normals.

Ex. 1. If the normal at the point $(at^2, 2at)$ on a parabola $y^2=4ax$ meets it again at $(at_1^2, 2at_1)$, prove that

$$t_1=-\left(t+\frac{2}{t}\right).$$

The equation to the line joining the points

$$(at^2, 2at), (at_1^2, 2at_1)$$

is
$$y-2at=\frac{2}{t+t_1}(x-at^2)\ldots(1).$$

But the normal to the parabola at $(at^2, 2at)$ is

$$y+tx=2at+at^3.$$

Since this is the same as (1),

$$-t=\frac{2}{t+t_1}$$

i.e.
$$t_1=-\left(t+\frac{2}{t}\right).$$

Ex. 2. Show that the locus of the poles of normal chords of $y^2=4ax$ is the curve

$$(x+2a)y^2+4a^3=0.$$

[U.P.C.S., 1940]

Let (x', y') be the pole of the normal chord

$$y=mx-2am-am^3\ldots(1).$$

The polar of (x', y') is

$$yy'=2ax+2ax'.$$

This must be identical with (1).

Comparing co-efficients,

$$\frac{1}{y'}=\frac{m}{2a}=-\frac{2am+am^3}{2ax'}.$$

Hence, $$\frac{1}{y'} = -\frac{a(2y'^2+4a^2)}{x'y'^3},$$

or $$y'^2(x'+2a)+4a^3=0.$$

Hence the locus of (x', y') is

$$y^2(x+2a)+4a^3=0.$$

8·41. Co-normal points.

The equation of the normal at the point $(am^2, -2am)$ of the parabola $$y^2=4ax$$

is $$y=mx-2am-am^3.$$

If this passes through (h, k),

$$k=mh-2am-am^3.$$

This being a cubic in m has three roots of which at least one must be real.

From a given point we can therefore draw three normals to a parabola, of which at least one must be real.

If m_1, m_2, m_3 be the roots of the above cubic,

$$m_1+m_2+m_3=0.$$

But the ordinates y_1, y_2, y_3 of the feet of these normals are $-2am_1$, $-2am_2$, $-2am_3$.

Hence, $$y_1+y_2+y_3=0.$$

That is, *the algebraic sum of the ordinates of the feet of the normals from any point to the parabola is zero.*

Ex. If the normals at two points of a parabola be inclined to the axis at the angles θ, φ such that $\tan\theta\tan\varphi=2$, show that they intersect on the parabola.

The equation to any normal to the parabola $y^2=4ax$ is

$$y=mx-2am-am^3.$$

If this passes through (h, k),

$$k=mh-2am-am^3.$$

Arranging in descending powers of m,

$$am^3+m(2a-h)+k=0.$$

If m_1, m_2, m_3 be the roots of this equation,

$$m_1+m_2+m_3=0 \quad \ldots(1),$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-h}{a} \quad \ldots(2),$$

Then

$$m_1m_2m_3=-\frac{k}{a} \quad \ldots(3).$$

Let m_1, m_2 be $\tan\theta$, $\tan\phi$ respectively.

Then

$$m_1m_2=2 \quad \ldots(4).$$

From (1), (2) and (4),

$$2-m_3^2=\frac{2a-h}{a} \quad \ldots(5).$$

From (3) and (4),

$$m_3=-\frac{k}{2a} \quad \ldots(6).$$

Eliminating m_3 between (5) and (6),

$$2-\frac{k^2}{4a^2}=\frac{2a-h}{a},$$

i.e.,

$$k^2=4ah,$$

or the point (h, k) lies on the parabola.

8·42. Circle through Co-normal points.

Let the normals at the points P, Q, R of the parabola $y^2=4ax$ meet in (h, k). The 'm's' of these normals are the roots of the equation

$$am^3+m(2a-h)+k=0.$$

Further, since ordinate y of any of the above points and the slope of the corresponding normal are related by the equation $$y=-2am,$$

the cubic giving the ordinates of P, Q and R is

$$y^3+4a(2a-h)y-8a^2k=0 \quad \dots (1)$$

Let the circle PQR be

$$x^2+y^2+2gx+2fy+c=0.$$

The y—co-ordinates of the points of intersection of this circle and the parabola are the roots of the equation

$$\frac{y^4}{16a^2}+y^2+2g\frac{y^2}{4a}+2fy+c=0,$$

i.e., $$y^4+y^2(16a^2+8ag)+32a^2fy+16a^2c=0 \quad \dots (2).$$

Three roots of (2) are the same as the three roots of (1).

Let y_1, y_2, y_3 be these three roots and y_4 the fourth root of (2).

Hence from (1),

$$y_1+y_2+y_3=0.$$

Also from (2),

$$y_1+y_2+y_3+y_4=0.$$

Therefore $$y_4=0.$$

Hence the circle PQR passes through the vertex of the parabola.

From the equation of the circle, we then have $c=0$.

Substituting for c in (2) and removing the common factor y

$$y^3+y(16a^2+8ag)+32a^2f=0 \quad \dots (3).$$

Equations (3) and (1) are now identical.

Comparing co-efficients,

$$16a^2+8ag=8a^2-4ah,$$

and

$$32a^2f=-8a^2k.$$

From these, $2g=-(h+2a)$, and $2f=-\dfrac{k}{2}$.

The equation to the *circle through the co-normal points* P, Q, R is

$$x^2+y^2-(h+2a)x-\tfrac{1}{2}ky=0.$$

Ex. 1. Prove that the chord of the parabola $y^2=4ax$ whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$ is a normal to the curve, and that its length is $6\sqrt{3}a$.

Ex. 2. Prove that the length of the intercept on the normal at the point $P\ (at^2, 2at)$ of a parabola $y^2=4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$.

Ex. 3. Prove that the locus of the middle points of normal chords of a parabola $y^2-4ax=0$ is

$$\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a.$$

Ex. 4. The normals at two points P, Q on the parabola $y^2=4ax$ intersect on the curve. Show that the ordinates of P, Q are the roots of the quadratic $y^2+ky+8a^2=0$. Show also that PQ passes through a fixed point on the axis of the parabola.

Ex. 5. PQ is a chord of a parabola normal at P, AQ is drawn from the vertex A; and through P a line is drawn parallel to AQ meeting the axis in R. Show that AR is double the focal distance of P.

Ex. 6. Prove that the chord of the parabola $y^2=4ax$ which is normal at the point whose abscissa is $2a$ subtends a right angle at the vertex.

Ex. 7. Prove that the locus of points such that two of three normals from them to the parabola $y^2=4ax$ coincide is

$$27ay^2=4(x-2a)^3.$$

As in § 8·41, the equation in 'm' of the normals passing through (h, k) is

$$am^3+m(2a-h)+k=0 \qquad \ldots \quad (1)$$

Since two of the normals from (h, k) coincide, the roots of (1) are m_1, m_1, m_2.

Hence,

$$2m_1+m_2=0 \qquad (2),$$

$$2m_1m_2+m_1^2=\frac{2a-h}{a} \qquad \ldots \quad (3),$$

$$m_1^2m_2=-\frac{k}{a} \qquad \ldots \quad (4),$$

Substituting $m_2=-2m_1$ in (3) and (4),

$$3m_1^2=-\frac{2a-h}{a}$$

$$2m_1^3=\frac{k}{a}.$$

Eliminating m_1 between these,

$$\left(-\frac{2a-h}{3a}\right)^3=\left(\frac{k}{2a}\right)^2,$$

i.e.,

$$27\,ak^2=4(h-2a)^3.$$

Hence the required locus is

$$27ay^2=4(x-2a)^3.$$

Ex. 8. If three normals from a point to the parabola $y^2=4ax$ cut the axis in points whose distances from the vertex are in arithmetical progression, show that the point lies on the curve

$$27ay^2 = 2(x-2a)^3.$$

8 5. Propositions on the parabola.

(1) *The tangent at any point of a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.*

Let P be any point $(at^2, 2at)$ on the parabola $y^2=4ax$. The equation of the tangent at P is

$$ty = x + at^2 \qquad \ldots \quad (1)$$

The equation to the focal chord through P is

$$y = \frac{2t}{t^2-1}(x-a) \qquad \ldots \quad (2)$$

The tangent of the angle which (1) makes with (2) is

$$\frac{\dfrac{2t}{t^2-1} - \dfrac{1}{t}}{1+\dfrac{2}{t^2-1}} = \frac{1}{t},$$

which is the same as the slope of the tangent.

Hence the proposition.

(2) *The portion of a tangent to a parabola intercepted between the directrix and the curve subtends a right angle at the focus.*

Let P be a point 't' on the parabola $y^2=4ax$. The equation of the tangent at P is

$$ty = x + at^2.$$

This meets the directrix $x+a=0$ in $\left(-a, \dfrac{at^2-a}{t}\right)$

The slope of the line joining this point to the focus $(a, 0)$ is $-\frac{1-t^2}{2t}$, and the slope of the focal chord through P is $\frac{2t}{t^2-1}$.

The product of the two slopes is -1, which proves the proposition.

(3) *The tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.*

Let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2=4ax$.

The co-ordinates of the point of intersection of tangents at these points are $\{at_1t_2, a(t_1+t_2)\}$.

The equation to the chord is

$$\tfrac{1}{2}(t_1+t_2)\,y=x+at_1t_2.$$

Since this passes through the focus $(a, 0)$,

$$a+at_1t_2=0$$

i.e., $$t_1t_2=-1 \qquad \dots \quad (1)$$

Hence the x-co-ordinate of the point of intersection of the tangents is $-a$, or the tangents intersect on the directrix of the parabola.

Further the slopes of the tangents at the two points are

$$\frac{1}{t_1},\ \frac{1}{t_2}.$$

From (1) therefore, the angle between the tangents is a right angle.

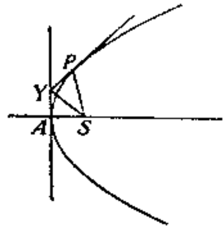
This proves the proposition.

(4) *If SY be perpendicular to the tangent at a point P of a parabola, then Y lies on the tangent at the vertex and $SY^2=AS.SP$.*

Let P be a point $(at^2, 2at)$ on the parabola $y^2=4ax$.

The equation of the tangent at P is

$$ty = x + at^2 \qquad \dots \quad (1)$$



The equation of the perpendicular from $S\ (a, 0)$ on this tangent is

$$t\,(x-a)+y=0,$$

or

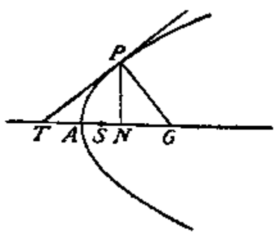
$$tx+y=at \qquad \dots \quad (2)$$

The x—co-ordinate of the point of intersection of (1) and (2) is evidently zero.

Hence Y lies on the tangent at the vertex.

Also,
$$SY^2 = a^2\ (1+t^2)$$
$$= a.(a+at^2)$$
$$= AS.SP. \qquad \text{(Ex. 1, §8·12)}$$

(5) *The subtangent of a point on a parabola is bisected at the vertex and the subnormal is of constant length.*



Definitions. If the tangent and normal at a point P of a parabola meet the axis in T and G respectively, and if PN be the ordinate of P, then NT is called the *Subtangent* and NG the *Subnormal* of P.

The equation of the tangent at P $(at^2, 2at)$ is

$$ty = x + at^2.$$

This meets the axis in T of which the co-ordinates are obviously $(-at^2, 0)$.

Thus $$AT = at^2 = AN.$$

Hence the subtangent is bisected at the vertex.

The equation of the normal at P is

$$tx + y = 2at + at^3.$$

Therefore, $$AG = 2a + 2at^2.$$

Hence $$NG = AG - AN = 2a.$$

The subnormal is thus equal to the semi-latus rectum of the parabola.

(6) *The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.*

The tangents at the points 't_1', 't_2', 't_3', of the parabola $y^2 = 4ax$ are

$$t_1 y = x + at_1^2 \quad \ldots \quad (1),$$

$$t_2 y = x + at_2^2 \quad \ldots \quad (2),$$

$$t_3 y = x + at_3^2 \quad \ldots \quad (3).$$

The point of intersection of (1) and (2) is

$$\{at_1t_2,\ a\ (t_1 + t_2)\}$$

The equation to the perpendicular from this point on (3) is

$$t_3(x - at_1t_2) + y - a\ (t_1 + t_2) = 0,$$

or $$t_3 x + y = a\ (t_1 + t_2 + t_1t_2t_3) \quad (4)$$

Similarly the equation of the perpendicular on (1) from the point of intersection of (2) and (3) is

$$t_1x+y=a(t_2+t_3+t_1t_2t_3) \qquad \dots \quad (5).$$

From (4) and (5), the co-ordinates of the orthocentre are

$$\{-a,\ a(t_1+t_2+t_3+t_1t_2t_3)\},$$

which accordingly lies on the directrix.

Ex. 1. Prove that the circle described on any focal chord of a parabola as diameter touches the directrix.

If $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2=4ax$, we have $t_1t_2=-1$.

The equation of the circle described on this chord as diameter is

$$(x-at_1^2)(x-at_2^2)+(y-2at_1)(y-2at_2)=0,$$

or $$x^2+y^2-ax(t_1^2+t_2^2)-2ay(t_1+t_2)+a^2t_1^2t_2^2+4a^2t_1t_2=0,$$

or $$x^2+y^2-ax(t_1^2+t_2^2)-2ay(t_1+t_2)-3a^2=0.$$

Putting $x=-a$,

$$y^2+a^2(t_1^2+t_2^2)-2ay(t_1+t_2)-2a^2=0,$$

or $$y^2+a^2\{(t_1+t_2)^2-2t_1t_2\}-2ay(t_1+t_2)=0,$$

or $$y^2-2ay(t_1+t_2)+a^2(t_1+t_2)^2=0,$$

or $$\{y-a(t_1+t_2)\}^2=0.$$

The directrix therefore meets the circle in two coincident points. Hence it is a tangent to the circle.

Ex. 2. O is the pole of a chord PQ of a parabola; prove that the perpendiculars from P, O, Q on any tangent to the curve are in geometrical progression.

Ex. 3. A chord is a normal to a parabola and is inclined at any angle θ to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3\theta \operatorname{cosec}^3\theta$.

Ex. 4. Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

Hint. The tangents at the points 't_1', 't_2', 't_3' of the parabola $y^2=4ax$ form the triangle whose vertices are the points.

$$\{at_1t_2, a(t_1+t_2)\} \text{ etc.}$$

Determine g, f and c so that the circle

$$x^2+y^2+2gx+2fy+c=0$$

circumscribes the triangle.

The equation will be found to be

$$x^2+y^2-ax(1+t_1t_2+t_2t_3+t_3t_1)-ay(t_1+t_2+t_3-t_1t_2t_3)$$
$$+a^2(t_1t_2+t_2t_3+t_3t_1)=0.$$

Ex. 5. A circle passes through the focus of a parabola. The tangents to the parabola from a point of the circle cut the circle again in points P,Q. Prove that PQ touches the parabola. [London, 1945]

8·6. Parallel Chords. If (x', y') is the middle point of a chord of the parabola $y^2=4ax$, its equation is

$$yy'-2ax=y'^2-2ax'$$

If the chord is parallel to $y=mx+c$,

$$\frac{2a}{y'}=m$$

Hence the locus of the middle points of chords of the parabola $y^2=4ax$ drawn parallel to the line $y=mx+c$ is

$$\mathbf{y}=\frac{\mathbf{2a}}{\mathbf{m}},$$

which is parallel to the axis of the parabola.

Any line drawn parallel to the axis of a parabola is called a *diameter*.

Thus a *diameter of a parabola bisects a system of parallel chords which are called the ordinates of that diameter.*

8·61. Tangent at the extremity of a diameter. The equation of the diameter of the parabola $y^2=4ax$ which bisects chords parallel to $y=mx+c$ is

$$y=\frac{2a}{m}.$$

This meets the parabola in the point whose co-ordinates are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

The tangent at this point is

$$y=mx+\frac{a}{m}$$

which is parallel to the given system of chords.

Hence *the tangent at the extremity of a diameter of a parabola is parallel to the chords which that diameter bisects.*

This could be proved more easily by considering the fact that at the extremity of the diameter the chord which is bisected by that diameter becomes a tangent to the parabola.

8·62. Tangents at the ends of a chord of a parabola. The slope 'm' of the chord joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ of the parabola $y^2=4ax$ is $\dfrac{2}{t_1+t_2}$.

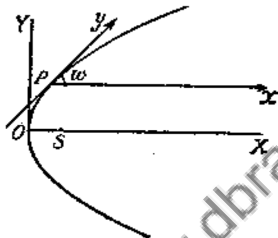
Hence the diameter bisecting this chord is

$$y=a(t_1+t_2).$$

But $a(t_1+t_2)$ is the $y-$co-ordinate of the point of intersection of the tangents at $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

Hence *the tangents at the extremities of any chord of a parabola meet on the diameter which bisects the chord.*

8·63. Equation of Parabola when the axes are a diameter and the tangent at the extremity of that diameter.



Let P be a point $(at^2, 2at)$ on the parabola $y^2=4ax$.

Transferring the origin to P retaining the direction of the axes, the equation of the parabola becomes

$$(y+2at)^2=4a(x+at^2).$$

Now let the new y-axis be rotated such that it coincides with the tangent at P. The new co-ordinates (X, Y) of any point on the parabola are connected with (x, y) the co-ordinates of the same point referred to rectangular axes through P by the equations (§ 3.3)

$$x=X+Y\cos\omega$$

$$y=Y\sin\omega,$$

where ω is the inclination of the tangent to the diameter.

The equation of the parabola referred to the tangent at P and the diameter through P is therefore

$$(Y\sin\omega+2at)^2=4a(X+Y\cos\omega+at^2),$$

that is $\quad Y^2\sin^2\omega+4aY(t\sin\omega-\cos\omega)=4aX.$

Now, the slope of the tangent at P is $\tan\omega = \frac{1}{t}$

Substiituting this value of tan ω the equation of the parabola can be written as

$$Y^2 = 4a(1+t^2)X$$

But $$a(1+t^2) = PS = b \text{ say.}$$

The equation of the parabola when the tangent at P and a diameter through P are taken as co-ordinate axes is thus

$$\mathbf{y^2 = 4bx.} \qquad \dots \quad (1)$$

It will thus be seen that *the equation $y^2 = 4ax$ is a particular case of the equation of the parabola referred to the tangent at any point and the corresponding diameter.*

From equation (1) we see that corresponding to any value of x there are two equal and opposite values of y. This therefore confirms the fact that chords parallel to a tangent are bisected by the diameter drawn through the point of contact of the tangent.

Ex. 1. Show that the normals at the extremities of a system of parallel chords of a parabola intersect upon a fixed line which is a normal to the parabola.

If y_1, y_2 be the ordinates of the extremities of a chord of the parabola $y^2 = 4ax$, drawn parallel to $y = mx + c$, $y_1 + y_2 = \frac{4a}{m}$. If y_3 be the ordinate of a third point on the parabola such that the normals at these three points are concurrent, $y_3 = -\frac{4a}{m}$, which fixes the third point. Hence the normals at the extremities of a system of parallel chords of the parabola intersect upon a fixed normal.

Ex. 2. P, Q, and R are three points on a parabola and the chord PQ cuts the diameter through R in V. Ordinates

PM and QN are drawn to this diameter. Prove that $RM.RN = RV^2$.

Ex. 3. The ordinates of three points on a parabola are in geometrical progression. Show that the tangents at the first and third points meet on the ordinate of the second point.

Ex. 4. If the diameter through any point O of a parabola meet any chord in P, and the tangents at the ends of that chord meet the diameter in Q, Q'; show that $OP^2 = OQ.OQ'$.

8·7. Equation of the parabola referred to two tangents. In the preceding article we have seen that the equation

$$y^2 = 4bx$$

represents a parabola whether the axes be oblique or rectangular.

If the axes are now transformed in any manner, the the new origin being (h, k), the new x-axis inclined at α to the original axis and the angle between the new axes being ω', we write for x and y respectively (§ 3.4)

$$x \frac{\sin(\omega - \alpha)}{\sin \omega} + y \frac{\sin\{\omega - (\alpha + \omega')\}}{\sin \omega} + h$$

and $$x \frac{\sin \alpha}{\sin \omega} + y \frac{\sin(\alpha + \omega')}{\sin \omega} + k,$$

where ω is the angle between the original axes.

The equation of the parabola then has the form

$$(Ax + By)^2 + 2gx + 2fy + c = 0. \qquad \dots (1)$$

This meets the axis of x in points whose abscissae are the roots of the equation

$$A^2x^2 + 2gx + c = 0.$$

If the parabola touches the axis of x at a distance a from the origin,

$$g^2 = A^2c, \quad -g = A^2a$$

Similarly, if the parabola touches the axis of y at a distance b from the origin,

$$f^2=B^2c, \quad -f=B^2b.$$

Therefore, $\quad B^2b^2=A^2a^2=c,$

from which, $\quad B=\pm A\,\dfrac{a}{b}$ ... (2)

Taking the negative sign,

$$B=-\frac{Aa}{b},\ g=-A^2a,\ f=-\frac{A^2a^2}{b},\ c=A^2a^2.$$

Substituting in (1),

$$\left(x-\frac{ay}{b}\right)^2-2ax-\frac{2a^2}{b}y+a^2=0,$$

or $$\left(\frac{x}{a}-\frac{y}{b}\right)^2-2\left(\frac{x}{a}+\frac{y}{b}\right)+1=0,$$

or $$\left(\frac{x}{a}+\frac{y}{b}\right)^2-2\left(\frac{x}{a}+\frac{y}{b}\right)+1=\frac{4xy}{ab},$$

or $$\frac{x}{a}+\frac{y}{b}-1=\pm 2\sqrt{\frac{xy}{ab}},$$

or $$\left(\sqrt{\frac{x}{a}}\mp\sqrt{\frac{y}{b}}\right)^2=1,$$

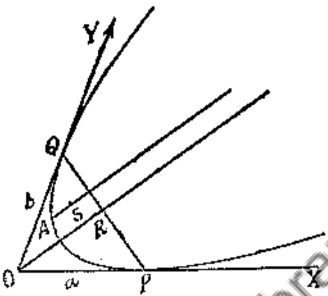
or $$\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}+\sqrt{\frac{\mathbf{y}}{\mathbf{b}}}=\mathbf{1} \qquad \text{... (3).}$$

If we had taken the positive sign in (2), equation (1) would reduce to

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=0.$$

which represents a pair of coincident straight lines.

Note. The radical signs in (3) can be either positive or negative. Thus in the annexed figure the abscissa of any point on the portion PAQ is $<a$, and the ordinate $<b$. For this portion of the parabola both signs are



positive. For points beyond P, the abscissa is $>a$, and the ordinate $<b$. The equation representing the portion beyond P is

$$\sqrt{\frac{x}{a}}-\sqrt{\frac{y}{b}}=1.$$

Similarly, the equation representing the portion beyond Q is

$$-\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

There is no part of the curve corresponding to two negative signs.

8·71. Focus of the parabola

$$\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}+\sqrt{\frac{\mathbf{y}}{\mathbf{b}}}=\mathbf{1}.$$

By working in rectangular co-ordinates it can be easily proved that if the tangents at P and Q to a parabola meet in O, as in the figure of the preceding article, then (i) the angles OSP and OSQ are equal, (ii) $OS^2=SP.SQ$.

From these two properties it follows that the triangles OSP and OSQ are similar. The angles SOP and SQO are therefore equal.

If now a circle is drawn through O, S, and Q, the line OP will be a tangent to it at O.

The focus S thus lies on the circle which passes through (o, b) and touches the axis of x at the origin.

The equation of this circle is

$$x^2+y^2+2xy \cos \omega = by,$$

the angle POQ being ω

The focus S also lies on the circle through P and touching the axis of y at the origin.

This circle is

$$x^2+y^2+2xy \cos \omega = ax.$$

The co-ordinates of the **focus,** which is the point of intersection of the two circles, are

$$\left(\frac{ab^2}{a^2+2ab \cos \omega+b^2}, \frac{a^2b}{a^2+2ab \cos \omega+b^2}\right).$$

8·72. Axis of the parabola

$$\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}+\sqrt{\frac{\mathbf{y}}{\mathbf{b}}}=\mathbf{1}.$$

Since the tangents at the ends of a chord of a parabola meet on the diameter which bisects the chord, the line through O and the middle point R of PQ will be parallel to the axis AS of the parabola. The co-ordinates of R are

$$\left(\frac{a}{2}, \frac{b}{2}\right).$$

The equation to OR is $y=\frac{b}{a}x$

The equation to the **axis,** which is a parallel line through S is therefore

$$ay-bx=\frac{ab\,(a^2-b^2)}{a^2+2ab\cos\omega+b^2}.$$

8·73. Vertex of the parabola

$$\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}+\sqrt{\frac{\mathbf{y}}{\mathbf{b}}}=\mathbf{1}.$$

The vertex is the point of intersection of the axis

$$ay-bx=\frac{ab\,(a^2-b^2)}{a^2+2ab\cos\omega+b^2}\quad\ldots\quad(1),$$

and the parabola $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$

Writing the equation to the parabola as

$$\left(\frac{x}{a}-\frac{y}{b}+1\right)^2=\frac{4x}{a},$$

we get on solving this and (1),

$$x=\frac{ab^2\,(b+a\cos\omega)^2}{(a^2+2ab\cos\omega+b^2)^2},$$

$$y=\frac{a^2b\,(a+b\cos\omega)^2}{(a^2+2ab\cos\omega+b^2)^2},$$

which are the co-ordinates of the **vertex.**

8·74. Directrix of the parabola

$$\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}+\sqrt{\frac{\mathbf{y}}{\mathbf{b}}}=\mathbf{1}.$$

To find the equation of the directrix we shall use the property that perpendicular tangents meet on the directrix. If we can find points on OP and OQ from which the tangents to the parabola are respectively perpendicular to OP and OQ, the line joining these two points will be the directrix itself.

If the point on OP is $(g, 0)$, the line perpendicular to OP is

$$y = m(x-g), \quad \text{where } 1 + m\cos\omega = 0.$$

The equation is thus

$$x + y\cos\omega = g.$$

This touches the parabola

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

or

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 - \frac{2x}{a} - \frac{2y}{b} + 1 = 0,$$

if the roots of the quadratic

$$\left(\frac{g - y\cos\omega}{a} - \frac{y}{b}\right)^2 - \frac{2(g - y\cos\omega)}{a} - \frac{2y}{b} + 1 = 0$$

are equal.

The quadratic can be written as

$$y^2(a + b\cos\omega)^2 - 2y\,[bg(a + b\cos\omega) + ab(a - b\cos\omega)] + b^2g^2 - 2ab^2g + a^2b^2 = 0.$$

The roots are equal if

$$\{bg(a + b\cos\omega) + ab(a - b\cos\omega)\}^2 = (a + b\cos\omega)^2(b^2g^2 - 2ab^2g + a^2b^2)$$

i.e., if

$$4a^2g(a + b\cos\omega) = 4a^3b\cos\omega,$$

from which

$$g = \frac{ab\cos\omega}{a + b\cos\omega}.$$

The point $\left(\dfrac{ab\cos\omega}{a + b\cos\omega}, 0\right)$ is therefore on the directrix.

Similarly the point $\left(0, \frac{ab \cos \omega}{b+a \cos \omega}\right)$ is on the directrix.

The equation of the **directrix** is therefore

$$x(a+b \cos \omega)+y(b+a \cos \omega)=ab \cos \omega.$$

8·75. Latus rectum of the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

The latus rectum is twice the perpendicular distance from the focus

$$\left(\frac{ab^2}{a^2+2ab \cos \omega+b^2}, \frac{a^2b}{a^2+2ab \cos \omega+b^2}\right)$$

on the directrix

$$x(a+b \cos \omega)+y(b+a \cos \omega)=ab \cos \omega.$$

Using the result of Ex. 1 § 3·3, the **latus rectum** after a little simplification is

$$\frac{4a^2b^2 \sin^2\omega}{(a^2+2ab \cos \omega+b^2)^{3/2}}.$$

Ex. 1. Find the equation to the tangent at any point (x', y') of the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

The straight line joining the points (x', y'), (x'', y'') is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x').$$

The points (x', y'), (x'', y'') lie on the parabola, if

$$\sqrt{\frac{x'}{a}}+\sqrt{\frac{y'}{b}}=1=\sqrt{\frac{x''}{a}}+\sqrt{\frac{y''}{b}},$$

i.e., if $$\frac{\sqrt{y''}-\sqrt{y'}}{\sqrt{b}} = -\frac{\sqrt{x''}-\sqrt{x'}}{\sqrt{a}} \quad \ldots(1)$$

Writing the equation to the line as

$$y-y' = \frac{(\sqrt{y''}-\sqrt{y'})(\sqrt{y''}+\sqrt{y'})}{(\sqrt{x''}-\sqrt{x'})(\sqrt{x''}+\sqrt{x'})}(x-x'),$$

and using (1),

$$y-y' = -\sqrt{\frac{b}{a}} \cdot \frac{\sqrt{y''}+\sqrt{y'}}{\sqrt{x''}+\sqrt{x'}}(x-x').$$

In the limit when $y'' \to y'$ and $x'' \to x'$, this reduces to

$$y-y' = -\sqrt{\frac{b}{a}} \cdot \frac{\sqrt{y'}}{\sqrt{x'}}(x-x'),$$

or $$\frac{x}{\sqrt{ax'}} + \frac{y}{\sqrt{by'}} = \sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1,$$

which is the equation to the tangent at (x', y').

Ex. 2. A parabola, whose latus rectum is l, slides in contact with each of two fixed perpendicular straight lines. Determine the locus of its vertex referred to these lines as axes.

Ans. $4x^{2/3}y^{2/3}(x^{2/3}+y^{2/3}) = l^2$.

Ex. 3. A parabola touches two given straight lines; if its axis passes through the point (p, q), referred to these lines as co-ordinate axes, prove that the focus lies on the curve $x^2-y^2-px+qy=0$.

EXAMPLES ON CHAPTER VIII

1. If a parabola touches the axis of 'x' and 'y' at $(2\sqrt{2}, 0)$ and $(0, 2\sqrt{2})$ respectively, find its equation.

2. Show that the polar of any point on the circle

$$x^2+y^2-2ax-3a^2=0$$

with respect to the circle

$$x^2+y^2+2ax-3a^2=0$$

will touch the parabola

$$y^2+4ax=0.$$

3. From the point where any normal to the parabola $y^2=4ax$ meets the axis is drawn a line perpendicular to this normal; prove that this line always touches an equal parabola.

4. Find the locus of the intersection of normals at the ends of a focal chord of a parabola.

5. Find the locus of the point of intersection of two normals to the parabola $y^2=4ax$ which are at right angles to another.

6. Two equal parabolas have the same vertex and their axes are at right angles. Prove that the common tangent touches each at the end of its latus rectum.

7. Tangents drawn from a variable point T to the parabola $y^2=4ax$ form a triangle of constant area A with the tangent at the vertex. Show that the locus of T is the curve $$x^2(y^2-4ax)=4A^2.$$

8. Prove that, if the difference of the squares of the perpendiculars on a moving line from two fixed points is constant, the line touches a fixed parabola.

Hint. Take the middle point of the line joining the fixed points as the origin.

9. Show that the point of intersection of two tangents to a parabola, which intercept a constant length on a fixed tangent, lies on an equal parabola.

10. Show that if tangents be drawn to the parabola $y^2=4ax$ from a point on the line $x+4a=0$, their chord of contact will subtend a right angle at the vertex.

11. The normals to the parabola $y^2=4ax$ from a point P meet the axis in A, B, C. When B is the midpoint of AC, show that the locus of P is

$$27ay^2=2(x-2a)^3.$$

12. The normals at the extremities of a chord of the parabola $y^2=4ax$ meet on the parabola. Show that the locus of the middle point of the chord is the parabola

$$y^2=2a(x+2a).$$

13. Show that the locus of the middle point of a variable chord of the parabola $y^2=4ax$ such that the focal distances of its extremities are in the ratio 2 : 1 is

$$9(y^2-2ax)^2=4a^2(2x-a)(4x+a).$$

14. Tangents are drawn at the ends of a normal chord of the parabola $y^2=4ax$. Show that the locus of their point of intersection is the curve

$$(x+2a)y^2+4a^3=0.$$

15. A tangent to the parabola $y^2=-4bx$ meets the parabola $y^2=4ax$ at P, Q. Prove that the middle point of PQ lies on the parabola

$$y^2(2a+b)=4a^2x.$$

16. Prove that the middle point of the intercept made on a tangent to a parabola by the tangents at two points P and Q lies on the tangent which is parallel to PQ.

17. If Q, R are the feet of the normals drawn from a variable point P on the parabola $y^2=4ax$, prove that the locus of the point where QR meets the tangent at P is the curve

$$3xy^2+4a(x^2+y^2)+16a^2(x+a)=0.$$

18. Tangents are drawn to the parabola $y^2=4ax$ from the point (α, β). Show that the length of their chord of contact is

$$\{(\beta^2-4a\alpha)(\beta^2+4a^2)\}^{\frac{1}{2}}/a,$$

and that the corresponding normals intersect at the point

$$\left(2a-\alpha+\frac{\beta^2}{a}, -\frac{\alpha\beta}{a}\right).$$

19. Show that if a chord of the parabola $y^2=4ax$ touches the parabola $y^2=4bx$, the tangents at its extremities meet on the parabola $by^2=4a^2x$, and the normals on the curve $(4a-b)^3y^2=4b^2(x-2a)^3$.

20. Prove that the area of the triangle formed by the feet of the normals from (h, k) to the parabola $y^2=4ax$ is

$$\{4a(h-2a)^3-27a^2k^2\}^{\frac{1}{2}}$$

21. If the lengths of the tangents drawn from an external point to a parabola are a and b, and the angle between them is θ, prove that the parameter of the parabola is

$$\frac{4a^2b^2\sin^2\theta}{(a^2+b^2+2ab\cos\theta)^{3/2}}$$ [Dublin, 1945]

22. If two normals to the parabola $y^2=4ax$ make complementary angles with the axis, show that their point of intersection lies on one of the curves

$$y^2=a(x-a), \quad y^2=a(x-3a)$$

23. The normals at three points P, Q, R of the parabola $y^2=4ax$ meet in (h, k). Prove that the centroid of the triangle PQR lies on the axis at distance $\frac{2}{3}(h-2a)$ from the vertex.

24. Prove that the line $Ax+By+C=0$ will touch the parabola

$$(x-x')^2+(y-y')^2=\frac{(lx+my+n)^2}{l^2+m^2},$$

if $$(A^2+B^2)(lx'+my'+n)=2(Al+Bm)(Ax'+By'+C).$$

Through each point of the straight line $x=my+b$ is drawn a chord of the parabola $y^2=4ax$ which is bisected at the point; prove that the chord touches the parabola

$$(y+2am)^2=8a(x-b).$$

[Indian Audit & Accts. Service. 1940.]

25. Prove that the triangle formed by three normals to a parabola is to the triangle formed by the three corresponding tangents in the ratio

$$(m_1+m_2+m_3)^2 : 1,$$

where m_1 m_2, m_3 are the tangents of the angles which the normals make with the axis.

26. A chord LL' of a given circle has its midpoint at O and its pole at P; a parabola is drawn with its focus at O and its directrix passing through P: prove that the tangent to this parabola at any point where it meets the circle passes through either L or L'.

27. A triangle circumscribes the circle $x^2+y^2=a^2$, and two angular points lie on the circle $(x-2a)^2+y^2=2a^2$: prove that the third angular point lies on the parabola $y^2=a(x-\frac{3}{4}a)$. Prove also that the three curves have two real and two impossible tangents.

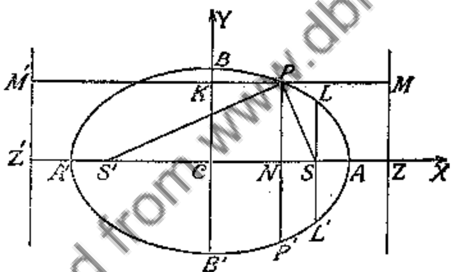
CHAPTER IX

THE ELLIPSE

9·1. Equation of the ellipse. *An ellipse is the locus of a point which moves such that its distance from a fixed point (the focus) is e (<1) times its distance from a fixed straight line (the directrix).*

e is called the *eccentricity* of the ellipse.

Let S be the *focus*, ZM the *directrix* and SZ perpendicular from S to the directrix.



The point A on SZ such that $AS = e.AZ$ clearly lies on the ellipse.

Since $e < 1$, there will be another point A' on ZS produced such that

$$SA' = eA'Z.$$

This point will also lie on the ellipse.

Now, $\quad AA' = AS + SA'$

$$= e(AZ + A'Z)$$

$= 2eCZ$, where C is the middle point of AA'.

If, therefore, $AA'=2a, \qquad CZ=\dfrac{a}{e}$.

Also, $\qquad SA'-AS=e(A'Z-AZ)=e.AA'=e.2a.$

But $\qquad SA'=SC+CA'$, and $AS=CA-CS$.

Therefore, $\qquad SC+CA'-(CA-CS)=e.2a,$

i.e., $\qquad 2CS=e.2a,$

or $\qquad CS=ae.$

Let C be the origin, and CA and a perpendicular CB as the axes of x and y respectively.

Let the co-ordinates of any point P on the ellipse be (x, y).

The equation to the directrix ZM is $x=\dfrac{a}{e}$ and the co-ordinates of the focus S are $(ae, 0)$.

By definition, $SP=e.PM$, PM being the perpendicular to the directrix.

This gives $\qquad (x-ae)^2+y^2=e^2.\left(\dfrac{a}{e}-x\right)^2,$

i.e., $\qquad x^2(1-e^2)+y^2=a^2(1-e^2),$

i.e., $\qquad \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1.$

Putting $a^2(1-e^2)=b^2$, this reduces to

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1} \qquad \dots (1)$$

which is the *standard equation* of the ellipse.

If in equation (1) we put $x=0, \qquad y=\pm b.$

The points B, B' in which the ellipse cuts the y-axis are thus at distance b from C.

9·11. Some properties of the ellipse. Since the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, contains only even powers of x and y, the ellipse represented by it is symmetrical with regard to both the co-ordinate axes. Also, any chord of the ellipse is bisected at the point C, for the points (x, y), $(-x, -y)$ simultaneously lie on the curve. The point C is therefore the *centre* of the ellipse.

The points A and A' are called the vertices, AA' $(=2a)$ is called the major axis, and BB' $(=2b)$ the minor axis of the curve. The major and minor axes are the axes of symmetry.

If $x>a$ numerically, y is imaginary, and if $y>b$ numerically, x is imaginary. The ellipse is therefore confined between the lines $x=\pm a$, $y=\pm b$, and is thus limited and closed.

The symmetry of the equation shows that there exists a second focus S' situated on the major axis at the same distance from C as S, and a second directrix $Z'M'$ for this focus cutting the major axis at right angles at Z' such that $CZ'=CZ$.

The line PN perpendicular to the major axis is called the *ordinate* of the point P. If PN produced meets the ellipse again in P', PNP' is called the *double ordinate* of P.

A double ordinate through a focus is called a *latus rectum*.

Since the co-ordinates of S are $(ae, 0)$, we get from the

equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$e^2+\frac{SL^2}{b^2}=1$$

or $$SL^2=b^2(1-e^2)=\frac{b^4}{a^2}$$

i.e., $$SL=\frac{b^2}{a}.$$

The *length of either latus rectum* is thus $\frac{2b^2}{a}$.

From the equation of the ellipse we easily infer that if the perpendicular distances p_1, p_2 of a moving point P from two perpendicular coplanar lines $u_1=0$, $u_2=0$ respectively are connected by the equation

$$\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}=1,$$

the point P describes an ellipse in the plane of the given lines provided a and b are real quantities. The centre of the ellipse is the point of intersection of the two lines and if $a>b$, the major axis lies along $u_2=0$ and is of length $2a$. The minor axis will be of length $2b$ lying along the line $u_1=0$.

Corollary. The equation $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a>b$ represents an ellipse of which the major axis lies along the y-axis and the minor axis along the x-axis. The foci are the points $(0, \pm ae)$.

Ex. 1. Obtain the equation to an ellipse whose focus is the point $(-1, 1)$, whose directrix is the line $x-y+3=0$ and whose eccentricity is $\frac{1}{2}$.

Ans. $7(x^2+y^2)+2xy+10(x-y)+7=0$.

Ex. 2. Find the latus retum, the eccentricity, and the co-ordinates of the foci of the ellipse $5x^2+4y^2=2$.

The equation can be written as $\frac{x^2}{\frac{2}{5}}+\frac{y^2}{\frac{1}{2}}=1$.

Since $\frac{1}{2}>\frac{2}{5}$, the major axis lies along the y-axis. Here $a^2=\frac{1}{2}$, $b^2=\frac{2}{5}$. The latus rectum$=\frac{2b^2}{a}=\frac{4\sqrt{2}}{5}$.

The eccentricity$=\sqrt{1-\frac{4}{5}}=\sqrt{1/5}$ and the co-ordinates of the foci are $\left(0, \pm\frac{1}{\sqrt{10}}\right)$.

Ex. 3. If the angle between the lines joining the foci of an ellipse to an extremity of the minor axis is 90°, find the eccentricity. Find also the equation of the ellipse if the major axis is $2\sqrt{2}$ units in length.

Ans. $\sqrt{\frac{1}{2}}$; $\frac{1}{2}x^2+y^2=1$.

Ex. 4. Find the centre and eccentricity of the ellipse

$$3(3x-2y+4)^2+2(2x+3y-5)^2=26.$$

Writing the equation as

$$3\left(\frac{3x-2y+4}{\sqrt{13}}\right)^2+2\left(\frac{2x+3y-5}{\sqrt{13}}\right)^2=2,$$

or $$\frac{\left(\frac{3x-2y+4}{\sqrt{13}}\right)^2}{\frac{2}{3}}+\left(\frac{2x+3y-5}{\sqrt{13}}\right)^2=1,$$

we see that the major axis is of length 2 with equation $3x-2y+4=0$, and the minor axis is of length $2\sqrt{\frac{2}{3}}$ with equation $2x+3y-5=0$. The co-ordinates of the centre which is the point of intersection of these two lines are $(-\frac{2}{13}, \frac{23}{13})$.

The eccentricity is $\sqrt{1-\frac{2}{3}}=\sqrt{\frac{1}{3}}$.

Ex. 5. Find the eccentricity and latus rectum of the ellipse $$3x^2+4y^2+6x-8y=5.$$

Ans. $\frac{1}{2}$; 3.

Ex. 6. Find the equation of an ellipse whose axes are of lengths 4 and 2 and their equations $x-y+3=0$ and $x+y-1=0$ respectively.

Ans. $5x^2-6xy+5y^2+22x-26y+29=0$.

9·12. A Geometrical Property. If PN be the ordinate of a point P on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$\frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1.$$

Therefore, $\frac{PN^2}{b^2} = 1 - \frac{CN^2}{a^2} = \frac{CA^2 - CN^2}{a^2} =$

Hence $\frac{PN^2}{AN.NA'} = \frac{BC^2}{AC^2}$.

Similarly, if PK be drawn perpendicular to axis,

$$\frac{PK^2}{BK.KB'} = \frac{AC^2}{BC^2},$$

for $\frac{PK^2}{a^2} + \frac{CK^2}{b^2} = 1$.

9·13. Sum of the focal distances of a point a point (x, y) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$PS = e.PM = e(CZ - CN) = a - ex.$$

And $PS' = e.PM' = e(CZ' + CN) = a + ex.$

Hence $PS + PS' = 2a.$

The sum of the focal distances of any point on is thus equal to the major axis.

This provides a means of tracing an ellipse. If a p moves on a piece of paper in contact with an inextens whose ends are fastened to two fixed points such th portions of the thread between it and the fixed always tight, the curve obtained will be an ellipse who be the fixed points.

9.14. Position of a point with regard to the

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The point (x', y') is inside, on or outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

according as the expression

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1$$

is negative, zero or positive.

The proof is obtained as for the parabola (§ 8·11) and may be easily worked out by the student.

9·15. Limiting Cases of the Standard equation of the ellipse.

Case I. If in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad b^2=a^2(1-e^2),$$

a remains constant and $e\to o$, $b^2\to a^2$. The foci then approach the centre and when e becomes nearly zero, the ellipse becomes almost a circle.

A circle is therefore the limiting case of an ellipse whose eccentricity tends to zero. The diameter of the circle is equal to the major axis of the ellipse.

Case II. If in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

the origin is transferred to the vertex $(-a, o)$, we get

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1,$$

i.e., $$\frac{x^2}{a^2}-\frac{2x}{a}+\frac{y^2}{b^2}=0,$$

i.e., $$\frac{x^2}{a}-2x+\frac{a}{b^2}y^2=0.$$

If now a and b both tend to infinity such that $\text{Lt}\,\frac{b^2}{a}$ is constant, k(say), the equation approximates to

$$y^2 = 2kx,$$

which is a parabola.

The centre of the ellipse in this limiting case lies at infinity.

Ex. 1. Prove that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Converting this into polar co-ordinates,

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

i.e.,

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

If r' be the length of the perpendicular semi-diameter,

$$r'^2 = \frac{a^2b^2}{b^2\cos^2\left(\theta+\frac{\pi}{2}\right) + a^2\sin^2\left(\theta+\frac{\pi}{2}\right)}$$

$$\therefore \quad \frac{1}{r^2} + \frac{1}{r'^2} = \frac{b^2+a^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Ex. 2. Discuss the position of the triangle formed by joining the points (1, 2), (3, −1) and (−2, −1) with respect to the ellipse

$$2x^2 + 3y^2 = 30 \qquad \textit{Ans.}\ \text{Inside the ellipse.}$$

Ex. 3. A straight line of given length has its extremities on two fixed straight lines which are at right angles. Show that the locus of any point on the line is an ellipse whose semi-axes are equal to the portions of the line in which the point divides it.

Ex. 4. A series of ellipses are described with a given focus and corresponding directrix; show that the locus of the extremities of their minor axes is a parabola.

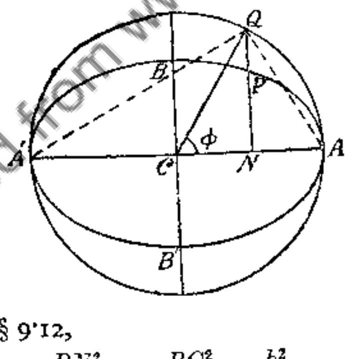
9·2. Auxiliary Circle. Def. *The circle described on the major axis of an ellipse as diameter is called the auxiliary circle of the ellipse.*

Let PN be the ordinate of any point P on the ellipse whose major axis is ACA' and minor axis BCB'.

Let NP produced meet the auxiliary circle in Q.

In the right angled triangle AQA',

$$QN^2 = AN.NA'$$



Also, from § 9·12,

$$\frac{PN^2}{AN.NA'} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2}.$$

Hence, $$\frac{PN^2}{QN^2} = \frac{b^2}{a^2},$$

that is $$\frac{PN}{QN} = \frac{b}{a}.$$

The point Q in which the ordinate of P meets the auxiliary circle is called the *corresponding point* to Q.

Ex. Perpendiculars are drawn upon a diameter from each point of a given circle. Show that the locus of the points which divide these perpendiculars in a given ratio is an ellipse.

9·21. Eccentric angle. Def. If the ordinate PN of a point P of an ellipse meets the auxiliary circle in Q, the angle NCQ between the major axis and the line CQ joining the centre C to Q is called the *eccentric angle* of P and is generally denoted by 'φ'.

We have $$CN = CQ \cos\varphi = a\cos\varphi,$$

and $$NP = \frac{b}{a}\,NQ = \frac{b}{a}\,CQ\sin\varphi = b\sin\varphi.$$

The co-ordinates of P are therefore $(\mathbf{a}\cos\varphi, \mathbf{b}\sin\varphi)$, and P is briefly called the point 'φ'.

9·22. Equation of the chord joining the points whose eccentric angles are θ and φ.

The equation is

$$y - b\sin\theta = \frac{b(\sin\theta - \sin\varphi)}{a(\cos\theta - \cos\varphi)}(x - a\cos\theta)$$

$$= -\frac{b}{a}\,\frac{\cos\dfrac{\theta+\varphi}{2}}{\sin\dfrac{\theta+\varphi}{2}}\,(x - a\cos\theta),$$

that is

$$\frac{y}{b}\sin\frac{\theta+\varphi}{2} + \frac{x}{a}\cos\frac{\theta+\varphi}{2} = \sin\theta\sin\frac{\theta+\varphi}{2}$$

$$+\cos\theta\cos\frac{\theta+\varphi}{2} = \cos\frac{\theta-\varphi}{2}.$$

Hence *the equation of the chord joining the points 'θ' and 'φ' on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is*

$$\frac{\mathbf{x}}{\mathbf{a}}\cos\frac{\theta+\varphi}{2}+\frac{\mathbf{y}}{\mathbf{b}}\sin\frac{\theta+\varphi}{2}=\cos\frac{\theta-\varphi}{2}.$$

If $\theta \to \varphi$, the chord becomes a tangent, and from the the above equation the tangent at the point 'φ' is

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1.$$

Ex. 1. Q is the point on the auxiliary circle corresponding to P on the ellipse. PLM is drawn parallel to the radius CQ to meet the axes in L and M. Prove that PM and PL are equal to the semi-axes.

Ex. 2. If θ, φ be the eccentric angles of a focal chord of an ellipse of eccentricity e, prove that

$$(i)\quad \pm e\cos\tfrac{1}{2}(\theta+\varphi)=\cos\tfrac{1}{2}(\theta-\varphi)\ ;$$

$$(ii)\quad \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\varphi+\frac{1\mp e}{1\pm e}=0.$$

Ex. 3. A point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose essentric angle is φ is joined to the foci S and S', and PS, PS' meet the curve again in Q and Q'. Show that the equation to QQ' is

$$\frac{x}{a}\cos\alpha\,(1-e^2)+\frac{y}{b}\sin\alpha\,(1+e^2)=e^2-1,$$

e being the eccentricity of the ellipse.

Ex. 4. If φ_1, φ_2, φ_3, φ_4 are the eccentric angles of the points in which an ellipse is cut by a circle, prove that

$$\varphi_1+\varphi_2+\varphi_3+\varphi_4=2n\pi$$

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The chords joining the points 'φ_1' 'φ_2', and 'φ_3', 'φ_4' are respectively

$$u_1 \equiv \frac{x}{a}\cos\frac{\varphi_1+\varphi_2}{2}+\frac{y}{b}\sin\frac{\varphi_1+\varphi_2}{2}-\cos\frac{\varphi_1-\varphi_2}{2}=0$$

and $$u_2 \equiv \frac{x}{a}\cos\frac{\varphi_3+\varphi_4}{2}+\frac{y}{b}\sin\frac{\varphi_3+\varphi_4}{2}-\cos\frac{\varphi_3-\varphi_4}{2}=0$$

The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda u_1 u_2=0$$

represents a conic through the four given points.

Since this is a circle, the co-efficient of xy in the above equation must be zero.

Hence,

$$\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_3+\varphi_4}{2}+\cos\frac{\varphi_3+\varphi_4}{2}\sin\frac{\varphi_1+\varphi_2}{2}=0,$$

that is $$\sin\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}=0,$$

that is $$\varphi_1+\varphi_2+\varphi_3+\varphi_4=2n\pi.$$

Ex. 5. If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1,$$

where $\alpha,\beta,\gamma,\delta$ are the eccentric angles of the extremities of the chords.

Ex. 6. Show that the area of a triangle inscribed in an ellipse is

$$\tfrac{1}{2}ab\{\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)\}$$
$$=-2ab\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta),$$

where α,β,γ are the eccentric angles of the angular points.

Ex. 7. A chord AP is drawn from the vertex of an ellipse of eccentricity e, along PA is taken a length PR equal to $PA \div e^2$, and RQ is drawn at right angles to the chord to meet the straight line through P parallel to the axis; prove that the locus of Q is a straight line perpendicular to the axis.

9·3. Equation of the tangent and other loci. As in Chapter V, or by regarding the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as a special case of the general equation of the second degree we have the following results :—

The equation of the tangent at (x',y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

At the point 'φ' this equation becomes

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1$$

Equation of the pair of tangents from (x', y') is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2.$$

Equation of the polar of (x', y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

This is also the equation of the chord of contact of tangents that can be drawn from (x',y') to the ellipse.

The equation of the chord whose middle point is (x',y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}.$$

9.4. Condition that the line $y=mx+c$ be a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The x—co-ordinates of the points in which the given line cuts the ellipse are the roots of the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

i.e.,
$$x^2(b^2+a^2m^2)+2a^2mcx+a^2(c^2-b^2)=0.$$

The line touches the ellipse if the roots of this quadratic are equal. This gives

$$a^4m^2c^2=(b^2+a^2m^2)(c^2-b^2)a^2,$$

i.e.,
$$c=\pm\sqrt{(a^2m^2+b^2)}.$$

The line $y=mx+\sqrt{(a^2m^2+b^2)}$ is thus a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

whatever be the value of m.

The student should find no difficulty in seeing that the *co-ordinates* of the point of contact are

$$\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\ \frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Since the equation $y=mx+\sqrt{(a^2m^2+b^2)}$ gives two values of m for fixed values of x and y, it follows that from a given point we can draw two tangents to the ellipse.

Further, substituting the two values of c in the equation $y=mx+c$, we see that for a given value of m there are two tangents $y=mx+\sqrt{a^2m^2+b^2}$ and $y=mx-\sqrt{a^2m^2+b^2}$. This shows that two tangents can be drawn to an ellipse parallel to a given line, and that these tangents are equidistant from the centre.

Cor. The line $y-k=m(x-h)+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

Ex. Show that the line $x\cos\alpha+y\sin\alpha=p$ touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

if

$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

9·5. The Director circle.

Def. *The director circle is the locus of points the tangents from which are at right angles.*

The equation of the pair of tangent from (x', y') to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2,$$

i.e.,

$$\frac{x^2}{a^2}\left(\frac{y'^2}{b^2}-1\right)+\frac{y^2}{b^2}\left(\frac{x'^2}{a^2}-1\right)-2xy\frac{x'y'}{a^2b^2}$$
$$+\text{first degree terms}=0.$$

These are at right angles if

$$\frac{1}{a^2}\left(\frac{y'^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x'^2}{a^2}-1\right)=0,$$

i.e.,

$$x'^2+y'^2=a^2+b^2$$

The equation of the *director circle* is therefore

$$\mathbf{x^2+y^2=a^2+b^2}.$$

Ex. 1. Prove that the tangents at the extremities of a diameter of an ellipse are parallel to one another.

Ex. 2. Show that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant is another ellipse.

The tangents at 'φ' and 'φ''are

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1,$$

and

$$\frac{x}{a}\cos\varphi'+\frac{y}{b}\sin\varphi'=1.$$

If the point of intersection of these tangents is (x', y'),

$$x'=a\frac{\cos\frac{1}{2}(\varphi+\varphi')}{\cos\frac{1}{2}(\varphi-\varphi')}, \text{ and } y'=b\frac{\sin\frac{1}{2}(\varphi+\varphi')}{\cos\frac{1}{2}(\varphi-\varphi')}.$$

Eliminating $\varphi+\varphi'$, the locus of (x', y') is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\tfrac{1}{2}(\varphi-\varphi').$$

Ex. 3. Chords of an ellipse are drawn through the positive end of the minor axis. Show that their middle points lie on another ellipse.

Ex. 4. Prove that the perpendicular from the focus of an ellipse whose centre is C on the polar of any point P will meet the line CP on the directrix.

Ex. 5. The ordinate MP of a point P on an ellipse is produced to meet the tangent at an end of the latus rectum through the focus F in R. Prove that $MR=FP$.

Ex. 6. Prove that the locus of the middle point of the portion of a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

included between the axes is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4.$$

Ex. 7. The tangent at a point P of an ellipse meets the auxiliary circle in two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is $(1+\sin^2\varphi)^{-\frac{1}{2}}$, where φ is the eccentric angle of P.

Ex. 8. An ellipse slides between two straight lines at right angles to one another. Show that the locus of its centre is a circle.

Ex. 9. Find the equations to the tangents at the ends of the latera recta of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and show that they pass through the intersections of the axis and the directrix.

Ex. 10. Show that the area of the triangle formed by the tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

at the points whose eccentric angles are α, β, γ respectively is

$$ab\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha)\tan\tfrac{1}{2}(\alpha-\beta).$$

Ex. 11. Prove that the common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.

The conic through the points of intersection of the lines

$$lx+my+n=0,$$
$$l'x+m'y+n'=0$$

and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(lx+my+n)(l'x+m'y+n')=0.$

If this is a circle, $lm'+ml'=0$,

that is $$\frac{l}{m}=-\frac{l'}{m'},$$

which proves the proposition.

Ex. 12. A circle is described on a chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

lying on the straight line

$$p\frac{x}{a}+q\frac{y}{b}=1$$

as diameter. Prove that the equation of the straight line joining the other two common points of the ellipse and the circle is

$$p\frac{x}{a}-q\frac{y}{b}=\frac{a^2+b^2}{a^2-b^2}.$$

Ex. 13. Tangents drawn from a point P to a given ellipse meet a given tangent whose point of contact is O in Q, Q'; prove that if the distance of P from the given tangent be constant, the rectangle $OQ.OQ'$ will be constant.

Ex. 14. From a point P of an ellipse two tangents are drawn to the circle on the minor axis. Prove that these tangents will meet the diameter at right angles to CP in points lying on two fixed straight lines parallel to the major axis.

Ex. 15. Two points P, Q are taken on an ellipse $x^2/a^2+y^2/b^2=1$ such that the perpendiculars from Q, P on the tangents at P, Q intersect on the ellipse; show that the locus of the pole of PQ is the ellipse

$$a^2x^2+b^2y^2=(a^2+b^2)^2.$$

9·6. The Normal.

The equation of the tangent at a point (x', y') of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

The equation of the normal which is perpendicular to the tangent at this point is

$$\frac{x'}{a^2}(y-y')-\frac{y'}{b^2}(x-x')=0,$$

i.e.,

$$-\frac{b^2y}{y'}+\frac{a^2x}{x'}=a^2-b^2.$$

At the point 'φ' this becomes

$$\mathbf{ax \sec \varphi - by \operatorname{cosec} \varphi = a^2 - b^2}.$$

9·61. Normals from a point to an ellipse.

The normal at the point $(a \cos \varphi, b \sin \varphi)$ of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$ax \sec \varphi - by \operatorname{cosec} \varphi = a^2-b^2.$$

If this passes through (h, k),

$$ah \sec \varphi - bk \operatorname{cosec} \varphi = a^2-b^2.$$

This equation will give the different values of φ for which the normals pass through (h, k).

Writing it as

$$\frac{ah\left(1+\tan^2\frac{\varphi}{2}\right)}{1-\tan^2\frac{\varphi}{2}}-\frac{bk\left(1+\tan^2\frac{\varphi}{2}\right)}{2\tan\frac{\varphi}{2}}=a^2-b^2,$$

or, on multiplying out,

$$bk\tan^4\frac{\varphi}{2}+2(ah+a^2-b^2)\tan^3\frac{\varphi}{2}+2(ah-a^2+b^2)\tan\frac{\varphi}{2}-bk=0.$$

This being a quartic equation gives four values of $\tan\frac{\varphi}{2}$.

If k_1 be one of the four values of $\tan\frac{\varphi}{2}$,

$$\tan\frac{\varphi}{2}=k_1,\ \varphi=2\tan^{-1}k_1$$

and the general value of $\varphi=2n\pi+2\tan^{-1}k_1$ which gives the same point on the ellipse as φ.

Thus corresponding to one value of $\tan\frac{\varphi}{2}$ we get one point on the ellipse, real or imaginary.

Hence *four normals can be drawn from a point to an ellipse.*

Ex. 1. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on the ellipse, such that the normals at them are concurrent, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

If the normals meet at (h, k), then $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$, $\tan\frac{\delta}{2}$ are by the preceding article the roots of the equation in t,

$$bkt^4+2(ah+a^2-b^2)t^3+2(ah-a^2+b^2)t-bk=0.$$

Therefore,

$$S_2\equiv\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=0,$$

$$S_4=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=-1.$$

But $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{S_1-S_3}{1-S_2+S_4}$, where $S_1=\Sigma\tan\frac{\alpha}{2}$, and $S_3=\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$.

Now, $1-S_2+S_4=0$, and $S_1\neq S_3$.

Hence $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\infty$,

that is $\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}=$ an odd multiple of $\frac{\pi}{2}$,

that is $\alpha+\beta+\gamma+\delta=$ an odd multiple of π.

Ex. 2. If the normals at the points whose eccentric angles are α, β, γ are concurrent, then

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

[Indian Audit & Accts. Service, 1947.]

If the normals at α, β, γ meet at (h, k) and if δ be the foot of the fourth normal from (h, k), then, as in the preceding example,

$$\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=0,$$

and $$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=-1.$$

Eliminating $\tan\frac{\alpha}{2}$ between these,

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$
$$-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-\cot\frac{\alpha}{2}\cot\frac{\gamma}{2}-\cot\frac{\beta}{2}\cot\frac{\gamma}{2}=0.$$

This gives

$$\Sigma\left\{\frac{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}-\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}}\right\}=0,$$

that is $$\Sigma\frac{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\alpha\sin\beta}=0,$$

that is $\Sigma \frac{\cos\alpha+\cos\beta}{\sin\alpha\,\sin\beta}=0,$

that is $(\cos\alpha+\cos\beta)\sin\gamma+(\cos\beta+\cos\gamma)\sin\alpha$
$+(\cos\gamma+\cos\alpha)\sin\beta=0,$

from which $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$

Ex. 3. Prove the converse of the above proposition, *viz.*, that if

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0,$$

the normals at α, β, γ will be concurrent.

9·7. Some propositions on the ellipse.

(1) *The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal distances of the point.*

Let P be any point (x', y') on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Let the tangent and normal at P meet the major axis in T and G respectively.

The equation of the normal PG is

$$a^2\frac{x}{x'}-b^2\frac{y}{y'}=a^2-b^2.$$

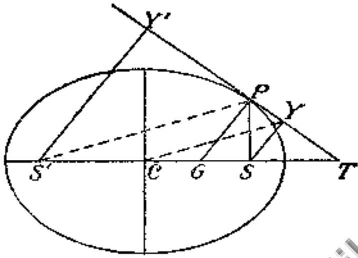
This meets the major axis in the point which is obtained by putting $y=0$ in the above equation.

$$\therefore \quad CG=\frac{x'}{a^2}(a^2-b^2)=x'\left(1-\frac{b^2}{a^2}\right)=e^2x'$$

$$\therefore \quad SG=CS-CG=ae-e^2x'=e(a-ex')$$
$$=e.PS. \quad (\S\ 9\cdot13)$$

Similarly $\qquad GS'=e.PS'.$

Hence
$$\frac{SP}{PS'}=\frac{SG}{GS'},$$



i.e., the normal PG bisects the angle SPS'.

Since the tangent is perpendicular to the normal, PT bisects the external angle between the focal distances of P.

(2) *If SY and $S'Y'$ be perpendiculars from the foci upon the tangent at any point P of the ellipse then Y and Y' lie on the auxiliary circle and $SY.S'Y'=b^2$.*

Let P be the point $(a\cos\varphi, b\sin\varphi)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the tangent at P is

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1 \qquad \dots \quad (1)$$

The equation to SY, which is perpendicular from $(ae, 0)$ to the tangent at P is

$$\frac{y}{a}\cos\varphi-\frac{(x-ae)}{b}\sin\varphi=0,$$

that is
$$\frac{x}{b}\sin\varphi-\frac{y}{a}\cos\varphi=\frac{ae}{b}\sin\varphi \qquad \dots \quad (2)$$

Since the point Y lies both on (1) and (2), the locus of Y, by squaring and adding (1) and (2) is

$$(x^2+y^2)\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)=\frac{b^2+a^2e^2\sin^2\varphi}{b^2}$$

$$=\frac{b^2+(a^2-b^2)\sin^2\varphi}{b^2}$$

$$=\frac{b^2\cos^2\varphi+a^2\sin^2\varphi}{b^2},$$

that is, $$x^2+y^2=a^2.$$

The point Y therefore lies on the auxiliary circle.

Similarly it may be proved that Y' lies on this circle.

Also, $$SY=\frac{1-e\cos\varphi}{\sqrt{\dfrac{\cos^2\varphi}{a^2}+\dfrac{\sin^2\varphi}{b^2}}}$$

$$=\frac{ab\,(1-e\cos\varphi)}{\sqrt{\cos^2\varphi.b^2+\sin^2\varphi.a^2}},$$

and $$S'Y'=\frac{ab(1+e\cos\varphi)}{\sqrt{\cos^2\varphi.b^2+\sin^2\varphi.a^2}}.$$

$$\therefore\quad SY.S'Y'=\frac{b^2(a^2-a^2e^2\cos^2\varphi)}{a^2\sin^2\varphi+b^2\cos^2\varphi}$$

$$=\frac{b^2\{a^2-(a^2-b^2)\cos^2\varphi\}}{a^2\sin^2\varphi+b^2\cos^2\varphi}$$

$$=\frac{b^2(a^2\sin^2\varphi+b^2\cos^2\varphi)}{a^2\sin^2\varphi+b^2\cos^2\varphi}$$

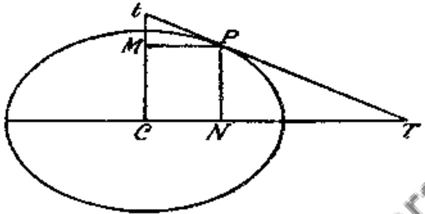
$$=b^2.$$

(3) *If the tangent at any point P meet the major and minor axes in T and t,*

$$CN.CT=a^2,\quad CM.Ct=b^2,$$

where N and M are the feet of the perpendiculars from P on the respective axes.

If P be the point (x', y') on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of the tangent at P is



$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

The coordinates of T are $(CT, 0)$.

Hence $$\frac{CT\,x'}{a^2}=1,$$

that is $$CT.CN=a^2.$$

Similarly $$Ct.CM=b^2.$$

Ex. 1. Perpendiculars SY and $S'Y'$ are dropped from the foci on any tangent to the ellipse. Prove that CY and CY' are parallel to $S'P$ and SP respectively.

Since $$CT=\frac{a^2}{CN},$$

$$S'T=CT+CS=\frac{a^2}{CN}+ae=\frac{a(a+eCN)}{CN}$$

$$\therefore \quad \frac{CT}{S'T}=\frac{a}{a+eCN}=\frac{CY}{S'P}.$$

Hence CY and $S'P$ are parallel. Similarly CY' and SP are parallel.

Ex. 2. Prove that the subtangent and subnormal of a point (x', y') on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are respectively $\left(\frac{a^2}{x'}-x'\right)$ and $\frac{b^2x'}{a^2}$.

Ex. 3. Show that the line $lx+my+n=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.

Hint. The normal to the ellipse $ax \sec\varphi-by \operatorname{cosec}\varphi=a^2-b^2$ should be identical with $lx+my+n=0$. Compare coeffs. etc.

Ex. 4. Show that the locus of poles of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

If (x', y') is the pole of the normal chord $ax \sec\varphi-by \operatorname{cosec}\varphi=a^2-b^2$, this equation must be identical with

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

Comparing co-efficients,

$$\frac{x'\cos\varphi}{a^3}=-\frac{y'\sin\varphi}{b^3}=\frac{1}{a^2-b^2},$$

that is $(a^2-b^2)\cos\varphi=\frac{a^3}{x'}$, $(a^2-b^2)\sin\varphi=-\frac{b^3}{y'}$.

Eliminating φ the required locus is

$$(a^2-b^2)^2=\frac{a^6}{x^2}+\frac{b^6}{y^2}.$$

Ex. 5. The length of the major axis intercepted between the tangent and normal at a point 'φ' on the ellipse is equal to the semi-major axis. Prove that the eccentricity of the ellipse is $\sqrt{\sec\phi(\sec\phi - 1)}$.

Ex. 6. Show that the eccentricity of the ellipse in which the normal at one end of a latus rectum passes through an end of the minor axis is given by the equation

$$e^4 + e^2 - 1 = 0.$$

Ex. 7. To any point P of an ellipse $b^2x^2 + a^2y^2 = a^2b^2$ corresponds a point Q on the auxiliary circle. The normals to the ellipse at P and the circle at Q meet in R. Prove that the locus of R is $x^2 + y^2 = (a+b)^2$.

Ex. 8. If p be the length of the perpendicular from a focus upon the tangent at any point P of the ellipse and r the distance of P from the focus, prove that

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

Ex. 9. If the normal at any point P cut the major axis in G, show that, for different positions of P, the locus of the middle point of PG will be an ellipse.

Ex. 10. If ABC be a maximum triangle inscribed in the ellipse, show that the eccentric angles of the vertices differ by $\frac{2\pi}{3}$ and that the normals at A, B, C are concurrent.

Let the eccentric angles of the points A, B, C, on the ellipse be α, β, γ; and let P, Q, R, be the three corresponding points on the auxiliary circle.

The areas of the triangles ABC, PQR are respectively

$$\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}, \text{ and } \frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}$$

The ratio of the two areas is $\frac{b}{a}$. The two triangles are therefore simultaneously greatest.

But the triangle PQR being in a circle is greatest when it is equilateral. In that case

$$\alpha \sim \beta = \beta \sim \gamma = \gamma \sim \alpha = \frac{2\pi}{3}.$$

Hence the triangle ABC is maximum when the eccentric angles of the vertices differ by $\frac{2\pi}{3}$.

Now $\alpha+\beta=2\alpha+\frac{2\pi}{3}$, $\beta+\gamma=2\alpha+\frac{6\pi}{3}$,

$$\gamma+\alpha=2\alpha+\frac{4\pi}{3}.$$

Hence $\quad \sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)$

$$=\sin\left(2\alpha+\frac{2\pi}{3}\right)+\sin 2\alpha+\sin\left(2\alpha+\frac{4\pi}{3}\right)$$

$$=\sin 2\alpha-\sin 2\alpha$$

$$=0,$$

which shows that the normals at A, B, C are concurrent.

Ex. 11. PQ is a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ parallel to a given line. Find the locus of the intersection of the normals at P and Q.

The normals at the points 'α', 'β' are

$$ax \sec\alpha - by \operatorname{cosec}\alpha = a^2-b^2,$$

and $$ax \sec\beta - by \operatorname{cosec}\beta = a^2-b^2.$$

The co-ordinates (x', y') of the point of intersection are given by

$$x' = \frac{a^2-b^2}{a}\cos\alpha\cos\beta\,\frac{\cos\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}} \quad \ldots (1)$$

$$y' = -\frac{a^2-b^2}{b}\sin\alpha\sin\beta\,\frac{\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}} \quad \ldots (2).$$

The equation to the chord PQ is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}.$$

Hence by the condition of the problem $\alpha+\beta=$const. $=2\theta$ say.

Now from (1) & (2),

$$\frac{ax'}{\cos\theta}+\frac{by'}{\sin\theta}=(a^2-b^2)\,\frac{\cos 2\theta}{\cos\dfrac{\alpha-\beta}{2}},$$

and

$$\frac{ax'}{\cos\theta}-\frac{by'}{\sin\theta}=(a^2-b^2)\,\frac{\cos(\alpha-\beta)}{\cos\dfrac{\alpha-\beta}{2}}$$

$$=(a^2-b^2)\left\{2\cos\frac{\alpha-\beta}{2}-\sec\frac{\alpha-\beta}{2}\right\}$$

Eliminating $\dfrac{\alpha-\beta}{2}$ between these equations, the locus of (x', y') is

$$a^2x^2+2abxy\,\text{cosec}\,2\theta+b^2y^2=(a^2-b^2)^2\cos^2 2\theta.$$

Ex. 12. Show that the rectangle under the perpendiculars drawn to the normal at a point P from the centre and from the pole of the normal is equal to the rectangle under the focal distances of P.

9·8. Conjuate Diameters. Two diameters of the ellipse which are such that each bisects chords parallel to the other are called conjugate diameters.

In Chapter V (§ 5.92) we obtained the condition satisfied by the conjugate diameters of the conic represented by the general equation of the second degree. We shall now obtain independently the relation between the slopes of two conjugate diameters of the ellipse.

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is bisected at the point (x', y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}$$

If this is parallel to the line $y=mx$,

$$-\frac{b^2}{a^2}\cdot\frac{x'}{y'}=m.$$

The locus of (x', y') is therefore the diameter

$$y=-\frac{b^2}{a^2m}\,x.$$

Writing it as $y=m'x$, we see that the diameter $y=m'x$ bisects chords of the ellipse parallel to the diameter $y=mx$, if

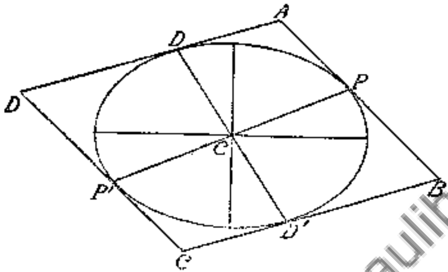
$$mm'=-\frac{b^2}{a^2}.$$

From the symmetry of this relation it is apparent, that the diameter $y=mx$ will at the same time bisect chords parallel to $y=m'x$.

The diameters $y=mx$, $y=m'x$ are therefore conjugate if

$$\mathbf{mm'=-\frac{b^2}{a^2}}.$$

9·81. The relation between the eccentric angles of an extremity of each of two conjugate diameters. Let θ and φ be the eccentric angles of the extremities P and D of two conjugate diameters.



The m's of CP and CD are

$$\frac{b}{a}\frac{\sin\theta}{\cos\theta} \text{ and } \frac{b}{a}\frac{\sin\varphi}{\cos\varphi}.$$

Hence

$$\frac{b}{a}\frac{\sin\theta}{\cos\theta}\cdot\frac{b}{a}\frac{\sin\varphi}{\cos\varphi}=-\frac{b^2}{a^2},$$

i.e., $\sin\theta\sin\varphi+\cos\theta\cos\varphi=0,$

i.e., $\cos(\theta-\varphi)=0,$

i.e., $\theta\sim\varphi=$ an odd multiple of $\dfrac{\pi}{2}$.

The co-ordinates of an extremity of each of two conjugate diameters can therefore be written as

$$(a\cos\varphi,\ b\sin\varphi),\ (-a\sin\varphi,\ b\cos\varphi).$$

9·82. The sum of the squares of conjugate semi-diameters.

If the co-ordinates of P are $(a\cos\varphi,\ b\sin\varphi)$, those of D are $(-a\sin\varphi, b\cos\varphi)$.

Therefore, $CP^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi$,

and $CD^2 = a^2 \sin^2 \varphi + b^2 \cos^2 \varphi$.

Then $CP^2 + CD^2 = a^2 + b^2$.

The sum of the squares of conjugate semi-diameters is constant and equal to the sum of the squares of the semi-axes of the ellipse, which are a particular case of conjugate semi-diameters.

9·83. The area of the parallelogram formed by the tangents at the extremities of two conjugate diameters.

The co-ordinates of the extremities P and D of the conjugate diameters PCP', DCD' can be written as $(a \cos \varphi, b \sin \varphi)$ and $(-a \sin \varphi, b \cos \varphi)$.

The tangent at P is

$$\frac{x}{a} \cos \varphi + \frac{y}{b} \sin \varphi = 1,$$

of which the slope is $-\frac{b}{a} \cot \varphi$. This being also the slope of CD, we see that the tangent at P is parallel to DCD'.

The tangent at P' $(-a \cos \varphi, -b \sin \varphi)$ will also be seen to be parallel to DCD'.

Similarly the tangents at D, D' will be parallel to PCP'.

The tangents at P, P', D, D' thus form the parallelogram $ABCD$, of which the area will be four times the area of the parallelogram $APCD$.

Now the area of the parallelogram $APCD$ is equal to $CD \times$ perpendicular from C upon the tangent at P

$$= \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} \cdot \frac{1}{\sqrt{\frac{\cos^2 \varphi}{a^2} + \frac{\sin^2 \varphi}{b^2}}}$$

$$= ab.$$

The area of the parallelogram $ABCD$ is then $4ab$.

The area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant and equal to the product of the axes.

9·84. Conjugate diameters a Special case of Conjugate lines. Since the polar of a point is the same as the chord of contact of tangents from it to ellipse, the pole of the diameter PCP' will be at the point of intersection of the tangents at P, P'. These tangents being parallel, the pole lies at infinity on the diameter DCD' which is parallel to each of the tangents at P and P'. The pole of the diameter DCD' similarly lies on PCP'. CP and CD are thus conjugate lines (Chapter V §5.72).

9.85. Equiconjugate diameters. Conjugate diameters which are equal to each other are called equi-conjugate.

The diameters CP, CD are equi-conjugate if

$$a^2\cos^2\varphi+b^2\sin^2\varphi=a^2\sin^2\varphi+b^2\cos^2\varphi,$$

which gives $\quad (a^2-b^2)\cos 2\varphi=0.$

Since $\quad a\neq b, \quad \cos 2\varphi=0,$

i.e., $$\varphi=\tfrac{1}{4}\pi \text{ or } \tfrac{3}{4}\pi.$$

Taking $\frac{1}{4}\pi$ for φ, the equation of CP is

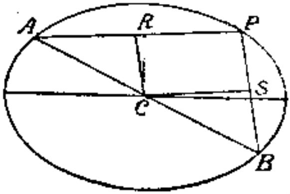
$$y=\frac{b}{a}\tan\varphi\, x$$

$$=\frac{b}{a}x.$$

The equation of CD is $y=-\dfrac{b}{a}x$.

The equi-conjugate diameters of an ellipse will thus be seen to be along the diagonals of the rectangle formed by the tangents at the ends of the major and minor axes.

9.86. Supplemental Chords. Def. The chords joining any point on an ellipse to the extremities of a diameter are called supplemental chords.

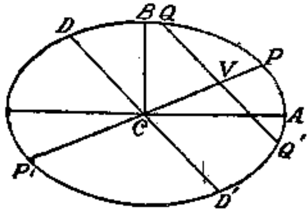


We shall show that supplemental chords are parallel to conjugate diameters.

Let ACB be a diameter of an ellipse of which the centre is at C. Let P be any point on the ellipse, and R, S the middle points of the chords AP, BP. The line CR which joins the middle points of AB and AP is parallel to BP. The diameter parallel to PB therefore bisects AP and in consequence all chords parallel to AP. Similarly the diameter along CS which is parallel to AP bisects all chords parallel to BP. This shows that CR, CS are portions of conjugate diameters.

The chords AP, BP are thus parallel to conjugate diameters.

9·87. Equation of the ellipse referred to two conjugate diameters.



The co-ordinates (x, y) of any point on the ellipse referred to CA and CB as axes are related to the co-ordinates (X, Y) of the same point referred to conjugate diameters CP, CD as axes by the equations

$$x=pX+qY,\ y=p'X+q'Y,$$

where p, q, p', q' depend on the angles ACP and PCD.

The equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has therefore the form $\qquad AX^2+2HXY+BY^2=1 \qquad \ldots\ (1),$

when the conjugate diameters are the co-ordinate axes.

If $CP=a'$, $CD=b'$, the points $(a', 0)$ and $(0, b')$ lie on the ellipse.

From (1) then $\qquad Aa'^2=1,\ i.e.,\ A=\frac{1}{a'^2},$

and $\qquad Bb'^2=1,\ i.e.,\ B=\frac{1}{b'^2}.$

Further, since the points (X, Y) and $(X, -Y)$ both lie on (1), $\qquad H=0.$

The equation of the ellipse referred to conjugate diameters of lengths $2a'$ and $2b'$ is thus

$$\frac{X^2}{a'^2}+\frac{Y^2}{b'^2}=1.$$

Corollary. The equation of the ellipse referred to two equiconjugate diameters is $X^2+Y^2=a'^2$.

Ex. 1. If PCP', DCD' be two conjugate diameters of an ellipse and if QVQ' be a double ordinate of the diameter CP, then $QV^2 : PV.VP'=CD^2 : CP^2$.

We have $\qquad \frac{CV^2}{CP^2}+\frac{QV^2}{CD^2}=1,$

therefore $\qquad \frac{QV^2}{CD^2}=1-\frac{CV^2}{CP^2}=\frac{(CV+CP)(CP-CV)}{CP^2}$

$$=\frac{VP'.PV}{CP^2}.$$

Ex. 2. QVQ' is a double ordinate of the diameter CP, and the tangent at Q meets CP in T. Prove that $CV.CT = CP^2$.

If Q be the point (x', y'), the tangent at Q is

$$\frac{Xx'}{a'^2} + \frac{Yy'}{b'^2} = 1.$$

Putting $Y=0$, $X = \frac{a'^2}{x'}$, *i.e.*, $CT = \frac{CP^2}{CV}$.

Ex. 3. Show that tangents at the extremities of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

The equations of tangents at an extremity of each of two conjugate diameters are

$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1,$$

$$-\frac{x}{a}\sin\varphi + \frac{y}{b}\cos\varphi = 1.$$

Squaring and adding,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2,$$

which is the required locus.

Ex. 4. P and D are the extremities of a pair of conjugate radii of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the middle point of PD is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2},$$

and that PD touches this ellipse.

The co-ordinates of P and D can be written as

$$(a \cos \varphi, b \sin \varphi), \ \left\{a \cos (\varphi+\pi/2), b \sin \left(\varphi+\frac{\pi}{2}\right)\right\}$$

The middle point (x, y) of PD is given by

$$x=\frac{a\{\cos \varphi+\cos (\varphi+\pi/2)\}}{2}=\frac{a}{\sqrt{2}} \cos \left(\varphi+\frac{\pi}{4}\right),$$

$$y=\frac{b\{\sin \varphi+\sin (\varphi+\pi/2)\}}{2}=\frac{b}{\sqrt{2}} \sin \left(\varphi+\frac{\pi}{4}\right).$$

Eliminating φ, the required locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\tfrac{1}{2} \qquad \ldots (1)$$

The equation to PD is

$$\frac{x}{a} \cos \left(\varphi+\frac{\pi}{4}\right)+\frac{y}{b} \sin \left(\varphi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Writing this as

$$\frac{x}{\frac{a}{\sqrt{2}}} \cos \left(\varphi+\frac{\pi}{4}\right)+\frac{y}{\frac{b}{\sqrt{2}}} \sin \left(\varphi+\frac{\pi}{4}\right)=1,$$

we see that this touches (1).

Ex. 5. CP, CQ are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the circles with CP and CQ as diameters intersect at R ; show that R lies on the curve

$$2\left(x^2+y^2\right)^2=a^2 x^2+b^2 y^2.$$

R will be the foot of the perpendicular from C upon PQ. The equation to PQ from the preceding example is

$$\frac{x}{a} \cos \left(\varphi+\frac{\pi}{4}\right)+\frac{y}{b} \sin \left(\varphi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \qquad \ldots (1)$$

If $CR = r$ and the angle which CR makes with the x axis $=\theta$, then (1) must be the same as

$$x \cos \theta + y \sin \theta = r$$

Comparing co-efficients,

$$\frac{\cos\left(\varphi + \frac{\pi}{4}\right)}{a \cos \theta} = \frac{\sin\left(\varphi + \frac{\pi}{4}\right)}{b \sin \theta} = \frac{1}{r\sqrt{2}},$$

i.e., $$\cos\left(\varphi + \frac{\pi}{4}\right) = \frac{a \cos \theta}{r\sqrt{2}},$$

$$\sin\left(\varphi + \frac{\pi}{4}\right) = \frac{b \sin \theta}{r\sqrt{2}}.$$

Squaring and adding,

$$a^2 \cos^2\theta + b^2 \sin^2\theta = 2r^2,$$

which is the locus of R in polar co-ordinates.

Converting into Cartesian co-ordinates,

$$2(x^2+y^2)^2 = a^2x^2 + b^2y^2.$$

Ex. 6. Show that the locus of the intersection of normals at the extremities of conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the curve $$2(a^2x^2+b^2y^2)^3 = (a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

Ex. 7. CP, CQ are conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ;$$

a tangent parallel to PQ meets CP, CQ at R, S ; show that R and S lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

Ex. 8. A pair of conjugate diameters is produced to meet the directrix. Show that the orthocentre of the triangle so formed is the focus.

Ex. 9. If any pair of conjugate diameters of an ellipse cut the tangent at a point P in T and T', show that

$$TP.\,PT'=CD^2,$$

where CD is the diameter conjugate to CP.

Hint. Take CP and CD as co-ordinate axes. The lines $y=mx$, $y=m'x$ will be conjugate diameters if

$$mm'=-b'^2/a'^2,\text{ where } CP=a',\ CD=b'.$$

Ex. 10. If PCP' and DCD' are a pair of conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and Q is any point on the circle

$$x^2+y^2=c^2,$$

show that $$PQ^2+DQ^2+P'Q^2+D'Q^2=2(a^2+b^2+2c^2).$$

Ex. 11. Through a fixed point P a pair of lines is drawn parallel to a variable pair of conjugate diameters of a given ellipse. The lines meet the principal axes in Q and R respectivly. Show that the middle point of QR lies on a fixed straight line.

9·9. Miscellaneous Propositions.

1. *Show that the feet of the normals from a point* (h, k)

to the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

lie on a conic which passes through (o, o) *and* (h, k).

The equation of the normal at a point (x', y') is

$$\frac{x'}{a^2}(y-y')-\frac{y'}{b^2}(x-x')=0.$$

Since this passes through (h, k),

$$\frac{x'}{a^2}(k-y')-\frac{y'}{b^2}(h-x')=0.$$

The locus of (x', y') is therefore the conic

$$xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-\frac{h}{b^2}y+\frac{k}{a^2}x=0 \qquad \text{(1)}$$

This will be seen to be a *rectangular hyperbola*, of which the properties will be discussed in the next chapter. The conic obviously goes through (o, o), (h, k).

2. *The normals to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the ends of the chords* $lx+my-1=0$ *and* $l'x+m'y-1=0$ *will be concurrent if* $a^2ll'=b^2mm'=-1$.

Let the normals at the points of intersection of the line $lx+my-1=0$ and the ellipse meet in (h, k). If the feet of the other two normals through (h, k) lie on $l'x+m'y-1=0$, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(lx+my-1)(l'x+m'y-1)=0$$

represents a conic through the feet of the normals.

It must be possible to find a value of λ for which this conic is the rectangular hyperbola represented by equation (1) of the preceding article. In that case the coeffs. of x^2 and y^2 and the constant term are zero simultaneously.

Hence,
$$\frac{1}{a^2}+\lambda ll'=0,$$

$$\frac{1}{b^2}+\lambda mm'=0,$$

$$-1+\lambda=0.$$

This gives $\quad a^2ll'=b^2mm'=-1.$

3. *If a triangle is inscribed in an ellipse and two of its sides are parallel to given straight lines, the envelope of the third side is another ellipse.*

Let the eccentric angles of the vertices A, B, C of the triangle be $\theta_1, \theta_2, \theta_3$. If AB, AC be parallel to given straight lines,

$$\theta_1+\theta_2=\text{constant}=2\alpha_1,$$

$$\theta_1+\theta_3=\text{constant}=2\alpha_2.$$

This gives $$\theta_2-\theta_3=2(\alpha_1-\alpha_2).$$

Now the equation to BC is

$$\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2}+\frac{y}{b}\sin\frac{\theta_2+\theta_3}{2}=\cos\frac{\theta_2-\theta_3}{2}$$

$$=\cos(\alpha_1-\alpha_2)$$

$$=k, \text{ say.}$$

The line BC then obviously touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k^2.$$

Definition. If a line is a tangent to a certain curve in all of the infinite positions which it can take up, the curve is called the *envelope* of the moving line.

4. *The co-ordinates of the centre of the circle through the points 'α', 'β', 'γ' of the ellipse are given by*

$$x=\frac{a^2-b^2}{4a}\{\Sigma\cos\alpha+\cos(\alpha+\beta+\gamma)\},$$

$$y=\frac{b^2-a^2}{4b}\{\Sigma\sin\alpha-\sin(\alpha+\beta+\gamma)\}.$$

The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ cuts the circle

$$x^2+y^2+2gx+2fy+c=0$$

in points whose eccentric angles are the roots of the equation in θ, *viz.*

$$a^2\cos^2\theta+b^2\sin^2\theta+2ga\cos\theta+2fb\sin\theta+c=0,$$

that is

$$\{(a^2-b^2)\cos^2\theta+2ga\cos\theta+c+b^2\}^2=4f^2b^2\sin^2\theta$$
$$=4f^2b^2-4f^2b^2\cos^2\theta.$$

If δ be the fourth value of θ, the sum of the roots of this equation in $\cos\theta$, *viz.*

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=-\frac{4ga}{a^2-b^2} \quad \cdots \quad (1)$$

Similarly

$$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=-\frac{4fb}{b^2-a^2} \quad \cdots \quad (2)$$

Also $$\alpha+\beta+\gamma+\delta=2n\pi \qquad \text{(Ex. 4 § 9·22).}$$

Eliminating δ from (1) and (2) with the help of this,

$$-g=\frac{a^2-b^2}{4a}\{\Sigma\cos\alpha+\cos(\alpha+\beta+\gamma)\},$$

$$-f=\frac{b^2-a^2}{4b}\{\Sigma\sin\alpha-\sin(\alpha+\beta+\gamma)\},$$

which is the result.

5. *If ω is the difference of the eccentric angles of two points on the ellipse the tangents at which are at right angles, then $ab\sin\omega=\lambda\mu$, where λ, μ are the semi-diameters parallel to the tangent at the points and a, b are semi-axes of the ellipse.*

Let P and Q be the points 'φ_1', 'φ_2' on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

such that $\varphi_2-\varphi_1=\omega$. The diameter parallel to the tangent at P cuts the ellipse in points one of which is $(-a\sin\varphi_1, b\cos\varphi_1)$. The length of the semi-diameter is

$$\sqrt{a^2\sin^2\varphi_1+b^2\cos^2\varphi_1}$$

Similarly the length of the other semi-diameter is

$$\sqrt{a^2 \sin^2 \varphi_2 + b^2 \cos^2 \varphi_2}.$$

Hence $\quad \lambda\mu = \sqrt{(a^2 \sin^2 \varphi_1 + b^2 \cos^2 \varphi_1)(a^2 \sin^2 \varphi_2 + b^2 \cos^2 \varphi_2)}$

$$= \sqrt{\{(a^2 \sin \varphi_1 \sin \varphi_2 + b^2 \cos \varphi_1 \cos \varphi_2)^2 + a^2b^2 (\sin \varphi_1 \cos \varphi_2 - \cos \varphi_1 \sin \varphi_2)^2\}}$$

Since the tangents at P and Q are at right angles,

$$a^2 \sin \varphi_1 \sin \varphi_2 + b^2 \cos \varphi_1 \cos \varphi_2 = 0.$$

Hence $\quad \lambda\mu = ab \sin (\varphi_1 - \varphi_2)$

$$= ab \sin \omega.$$

EXAMPLES ON CHAPTER IX

1. Prove that the line joining two points on an ellipse the difference of whose eccentric angles is constant, touches another ellipse.

2. Chords of an ellipse pass through a fixed point; prove that the locus of their middle points is an ellipse with its axes parallel to those of the original ellipse.

3. Obtain the locus of the foot of the perpendicular from the centre of an ellipse on any tangent.

4. Show that the equations of a pair of lines, which are at right angles and each of which passes through the pole of the other with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

may be written

$$lx + my + n = 0, \quad n(mx - ly) + lm(a^2 - b^2) = 0.$$

Show also that the product of the distances of such a pair of lines from the centre depends only on their direction and cannot exceed $\frac{a^2-b^2}{2}$. [Math. Tripos]

5. Prove that the least value of the intercept of a tangent to an ellipse between the axes is equal to the sum of the semi-axes.

6. If through a given point on an ellipse any two lines at right angles to each other be drawn to meet the curve, the line joining their extremities will pass through a fixed point on the normal.

7. Show that the locus of middle points of normal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2.$$

8. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines, show that the fourth side will also be parallel to a fixed straight line.

9. PCP' and DCD' are conjugate diameters of an ellipse, and φ is the eccentric angle of P. Prove that $\frac{1}{2}\pi-3\varphi$ is the eccentric angle of the point where the circle $PP'D$ again cuts the ellipse.

10. The tangent at any point P of an ellipse cuts the equiconjugate diameters in T, T'; show that the triangles TCP, $T'CP$ are in the ratio of $CT^2 : CT'^2$ where C is the centre of the ellipse.

11. A diameter PP' of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ being taken, the normal at P' intersects the ordinate at P in Q. Prove that the locus of Q is the ellipse

$$\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1.$$

12. Parallel lines through the foci of an ellipse meet the tangent at the vertex A in P, Q and the lines joining P, Q to the other vertex A' meet the circle on AA' as diameter in R and S. Prove that RS is a tangent to the ellipse.

13. The foci of an ellipse are S_1, S_2 and B is an extremity of the minor axis. The circle BS_1S_2 cuts the ellipse again in points P, Q. Prove that the lines PB, QB are normal to the ellipse at P, Q respectively.

[London, 1945]

14. Perpendiculars PM, PN are drawn from any point P on the equi-conjugate diameters of an ellipse. Prove that the perpendicular from P on its polar line bisects MN.

15. A point P on the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is joined to the ends of the major axis and the joining lines meet the ellipse again in Q, Q'. Prove that equation of QQ' is

$$(a^2+b^2)y \sin\theta + 2b^2x \cos\theta = 2ab^2,$$

where θ is the eccentric angle of the point on the ellipse to which P corresponds. If the ordinate to P meet QQ' in R, R is the point of contact of QQ' with its envelope.

16. The eccentric angles of two points P and Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are φ_1 and φ_2, prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is

$$4ab \operatorname{cosec}(\varphi_1-\varphi_2),$$

and hence that it is least when P and Q are at the end of conjugate diameters.

17. If p, q be the lengths of two tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at right angles to one another, prove

that

$$\frac{4(a^2+b^2)^3}{p^2+q^2} = \left\{ a^2+b^2+a^2b^2\left(\frac{1}{p^2}+\frac{1}{q^2}\right)\right\}^2.$$

18. In an ellipse whose axes are in the ratio $\sqrt{2}+1:1$, a circle whose diameter joins the ends of two conjugate diameters of the ellipse will touch the ellipse.

19. If the normals at the four points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, prove that

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4.$$

20. A parallelogram circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and two of its angular points are on the lines $x^2-h^2=0$; prove that the other two are on the conic

$$\frac{x^2}{a^2}+y^2\left(1-\frac{a^2}{h^2}\right)b^2-1=0.$$

21. If the orthocentre of the triangle formed by the two tangents which can be drawn from a point to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and the chord of contact, lies on the ellipse, find the locus of the point.

22. A triangle circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its centroid lies on the axis of x at a distance c from the centre ; prove that the angular points of the triangle lie on the conic

$$\frac{(x-3c)^2}{a^2}+\frac{y^2(a^2-9c^2)}{a^2b^2}=4.$$

23. Prove that the eccentricity of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is given by $$\frac{2\cot\omega}{\sin 2\theta}=-\frac{e^2}{\sqrt{1-e^2}},$$

where ω is one of the angles between the normals at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$.

24. The chord PQ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is such that the tangents at P and Q each pass through the pole of the other with respect to the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. If the eccentric angles of P and Q be θ and φ, show that

$$\frac{\alpha^2}{a^2}\cos\theta\cos\varphi+\frac{\beta^2}{b^2}\sin\theta\sin\varphi=1,$$

and that PQ touches the ellipse

$$\frac{x^2}{a^4}\left(a^2+\alpha^2\right)+\frac{y^2}{b^4}\left(b^2+\beta^2\right)=\frac{(a^2+\alpha^2)(b^2+\beta^2)}{a^2\beta^2+b^2\alpha^2}.$$

25. A variable point P on an ellipse of eccentricity e is joined to the foci S and S'. Prove that the incentre of the triangle PS_1S_2 lies on an ellipse of eccentricity

$$\sqrt{\frac{2e}{1+e}}.$$

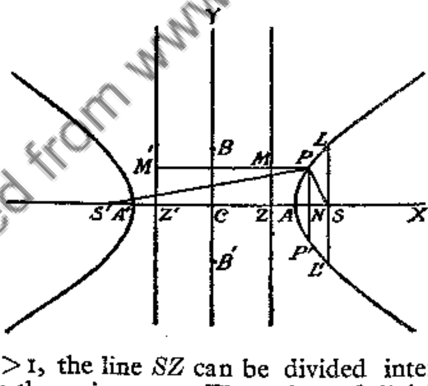
CHAPTER X

THE HYPERBOLA

10·1. Equation of the hyperbola. If a point moves such that its distance from a fixed point (the focus) is e (>1) times its distance from a fixed straight line (the directrix), its locus is a hyperbola.

For a hyperbola, therefore, the eccentricity e is a greater than unity.

Let S be the focus, ZM the directrix and SZ perpendicular from S on the directrix.



Since $e > 1$, the line SZ can be divided internally and externally in the ratio $e : 1$. The points of division A and A' clearly lie on the hyperbola.

Let $AA' = 2a$,

Since $$SA = e.AZ,$$

and $$SA' = e.A'Z,$$

we have $$SA + SA' = e\,(AZ + A'Z) = 2ae,$$

i.e., $$2SC=2ae, \text{ or } SC=ae,$$

where C is the middle point of AA'.

Also, $SA'-SA=e\,(A'Z-AZ)=e(AA'-2AZ)$,

i.e., $$2a=e.2(AC-AZ)=2e.CZ,$$

or $$CZ=\frac{a}{e}\ .$$

Now, let C be the origin, CS the x-axis and the perpendicular line CY the y-axis.

The co-ordinates of S are, therefore, $(ae, 0)$ and the equation to ZM is $x=\dfrac{a}{e}$.

If P be any point (x, y) on the hyperbola, we have $SP=e.PM$, PM being the perpendicular on ZM.

This gives

$$(x-ae)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2,$$

or $$x^2(e^2-1)-y^2=a^2(e^2-1),$$

i.e., $$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1.$$

Let $a^2(e^2-1)=b^2$. The equation then becomes

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2}=1 \qquad \dots (1),$$

which is the *standard form* of the equation of the hyperbola.

10·11. Some Properties of the hyperbola. The hyperbola is symmetrical about both the axes for the points (x, y), $(x, -y)$, $(-x, y)$, $(-x, -y)$ lie on the curve simultaneously. Also any chord of the hyperbola through C will be bisected at C, which is, therefore, the centre of the curve.

From equation (1) of the preceding article,

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1}.$$

If x is numerically less than a, y becomes imaginary.

The curve therefore does not lie between A and A'. If $x = \pm a$, $y = 0$ and for values of x numerically greater than a, y has two equal values, one positive and the other negative. If $x \to \pm\infty$ so does y.

The curve, therefore, consists of two infinite branches passing through A and A' respectively and lying one to the right of A, and the other to the left of A'.

If $x=0$, $y = \pm b\sqrt{-1}$. The intersections of the curve and the y-axis are thus imaginary.

The line AA' is called the **transverse axis.** The line BB', where B, B' are the points on the y-axis distant b from the centre, is called the **conjugate axis.** B and B', however, do not lie on the curve.

As in the case of the ellipse, there is a second focus S', $(-ae, 0)$ and a corresponding second directrix $Z'M'$ at the same distance from C as ZM.

The perpendicular PN to the transverse axis is called the **ordinate** of the point P and PNP' the **double ordinate.** A double ordinate through a focus is called a latus rectum.

The length of either latus rectum will be seen to be $\frac{2b^2}{a}$.

From the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$$\frac{CN^2}{CA^2} - \frac{PN^2}{CB^2} = 1,$$

i.e.,

$$\frac{CN^2}{CA^2} - 1 = \frac{PN^2}{CB^2},$$

i.e.,
$$\frac{AN.A'N}{CA^2}=\frac{PN^2}{CB^2},$$

i.e.,
$$PN^2 : AN.A'N :: CB^2 : CA^2,$$

which is the geometrical property expressed by the standard equation.

As in the case of the ellipse, the locus of a point P which moves in the plane of two perpendicular lines $u_1=0$, $u_2=0$ such that

$$\frac{{p_1}^2}{a^2}-\frac{{p_2}^2}{b^2}=1,$$

where p_1, p_2 are the perpendicular distances of P from $u_1=0$ and $u_2=0$ and a and b are both real is a hyperbola whose transverse axis lies along $u_2=0$ and is of length $2a$. The conjugate axis of the hyperbola lies along $u_1=0$ and is of length $2b$.

10·12. Focal distances of a point. In the figure of § 10·1 let P be any point (x, y) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, and PM' the perpendicular on the directrix $Z'M'$.

Now,
$$\begin{aligned}PS&=e.PM\\&=e.NZ\\&=e.(CN-CZ)\\&=ex-a,\end{aligned}$$

and
$$\begin{aligned}PS'&=ePM'\\&=eNZ'\\&=e(CN+CZ')\\&=ex+a.\end{aligned}$$

This gives
$$PS'-PS=2a.$$

The difference of the focal distances of any point of the hyperbola is constant and equal to the transverse axis.

10·13. Position of a point with regard to the hyperbola.

As in the case of the parabola (§ 8.11), the point (x', y') lies within, upon, or without the hyperbola according as the expression $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1$ is positive, zero or negative.

10·14. Rectangular hyperbola. If $b=a$, the hyperbola is called *rectangular*, or *equilateral*. The reason for this nomenclature is that the asymptotes, which we shall study in a subsequent article, are at right angles.

We have $$b^2=a^2(e^2-1),$$

i.e., $$e^2=1+\frac{b^2}{a^2}.$$

In the case of a rectangular hyperbola we thus have

$$e=\sqrt{2}.$$

10·15. Co-ordinates in terms of a Single Parameter. The co-ordinates of any point (x, y) on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

may be expressed as

$$x=a \sec \theta,\ y=b \tan \theta,$$

or $$x=a \cosh t,\ y=b \sinh t,$$

where $$\cosh t=\frac{e^t+e^{-t}}{2}, \text{ and } \sinh t=\frac{e^t-e^{-t}}{2}.$$

10·16. Limiting case of a hyperbola. The equation of the hyperbola when the transverse and conjugate axes are the coordinate axes is

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

or $$x^2-\frac{y^2}{e^2-1}=a^2$$

If in this e is kept constant, while $a \to o$, the hyperbola approximates to the pairs of lines

$$x^2 - \frac{y^2}{e^2-1} = 0,$$

the co-ordinates axes bisecting the angles between these lines.

A pair of straight lines is therefore the limiting case of a hyperbola whose axes are infinitesimal, while their ratio is finite.

Ex. 1. Find the equation to the hyperbola whose directrix is $2x+y=1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Ans. $7x^2+12xy-2y^2-2x+4y-7=0.$

Ex. 2. Find the lengths of the axes and the eccentricity of the hyperbola

$$4x^2-y^2-2y=3$$

Writing this equation as

$$4x^2-(y+1)^2=2$$

or

$$\frac{x^2}{\frac{1}{2}} - \frac{(y+1)^2}{2} = 1,$$

the transverse axis is of length $\sqrt{2}$, the conjugate axis of length $2\sqrt{2}$, and the eccentricity is $\sqrt{1+2.2} = \sqrt{5}$.

Ex. 3. In a rectangular hyperbola, prove that

$$SP.S'P = CP^2$$

Ex. 4. Find the equation of the hyperbola the lengths of whose transverse and conjugate axes are respectively $2\sqrt{2}$ and $2\sqrt{\frac{2}{3}}$, the equations of these axes being respectively $x-y+4=0$ and $x+y=0$.

Ans. $x^2-3xy+y^2+10x-10y+21=0.$

Ex. 5 On a level plain, the crack of the rifle and the thud of the ball striking the target are heard at the same instant. Show that the locus of the hearer is a hyperbola.

10·2. Loci connected with the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Since the equation of the hyperbola differs from that of the ellipse in having $-b^2$ for b^2, several of the results obtained for the ellipse hold good for the hyperbola when the sign of b^2 is changed.

We shall give below a few results for the hyperbola which may be obtained independently as for the general conic in Chapter V or deduced from those obtained for the ellipse.

(*i*) the tangent at any point (x', y') on the curve is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1,$$

which is also the equation of the chord of contact of the tangents from (x', y') and the polar of (x', y').

(*ii*) the normal at any point (x', y') on the curve is

$$\frac{x-x'}{\frac{x'}{a^2}}+\frac{y-y'}{\frac{y'}{b^2}}=0$$

(*iii*) the straight line $y=mx+\sqrt{a^2m^2-b^2}$ is a tangent to the curve for all values of m.

(*iv*) The straight line $x\cos\alpha+y\sin\alpha-p=0$ touches the curve if $p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$.

(*v*) The equation of the chord whose middle point is (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=\frac{x'^2}{a^2}-\frac{y'^2}{b^2}.$$

(*vi*) The diameters $y=mx$, $y=m'x$ are conjugate if

$$mm'=\frac{b^2}{a^2}.$$

(*vii*) The equation of the director circle is

$$x^2+y^2=a^2-b^2,$$

which is imaginary when $b>a$.

(*viii*) The equation of the pair of tangents from (x', y') is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2.$$

10·21. Tangent and Normal at the point $(a \sec\theta, b \tan\theta)$.

The equation of the tangent at $(a \sec\theta, b \tan\theta)$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1,$$

i. e.,
$$\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta.$$

The equation of the normal at this point is

$$\frac{\sec\theta}{a}\left(y-b\tan\theta\right)+\frac{\tan\theta}{b}\left(x-a\sec\theta\right)=0.$$

i.e.,
$$by\cot\theta+ax\cos\theta=a^2+b^2.$$

From this it can be seen that four normals can be drawn to a hyperbola from a point in its plane.

10·3. The Asymptotes. A straight line which touches a curve at infinity but does not lie wholly at infinity is called an *asymptote* of the curve.

Let us write the equation of the hyperbola as

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=1.$$

The straight line $\frac{x}{a}+\frac{y}{b}=k$ meets the hyperbola in points whose ordinates are given by

$$k\left(k-\frac{2y}{b}\right)=1.$$

Since any straight line meets a conic in two points, the coefficient of y^2 in the above equation being zero, one of the values of y is infinite. The other value is also infinite if $k=0$.

The straight line $\frac{x}{a}+\frac{y}{b}=0$ therefore meets the hyperbola in two coincident points at infinity and is an *asymptote* by definition.

The other asymptote will similarly be seen to be

$$\frac{x}{a}-\frac{y}{b}=0.$$

In the case of the rectangular hyperbola $x^2-y^2=a^2$, the asymptotes are $x=\pm y$. The angle between them is $90°$.

It will also be seen that the asymptotes of a hyperbola are the pair of tangents drawn from its centre.

The asymptotes of an ellipse are imaginary.

10·4. The Conjugate hyperbola. The hyperbola which has for its transverse and conjugate axes the conjugate and transverse axes of another hyperbola is called the 'conjugate hyperbola.'

Thus the hyperbola

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1 \qquad \ldots \quad (1),$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots \quad (2).$$

The two have the same asymptotes, *viz.*,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \qquad \ldots \quad (3).$$

It should be observed that the equation (3) differs from equation (2) by a constant, and that the equation (1) differs from equation (3) by the same constant.

Any transformation of axes will change the left hand members of (1), (2) and (3) in exactly the same way and the right hand constants after the transformation will differ in the same manner as before.

Hence the equation of the asymptotes differs from that of the hyperbola by a constant, and the equation of the conjugate hyperbola differs from that of the asymptotes by the same constant.

Ex. 1. Find the equation to the hyperbola conjugate to

$$x^2+3xy+2y^2+2x+3y+2=0.$$

The equation of the asymptotes differs from that of the hyperbola by a constant. Let it be, therefore,

$$x^2+3xy+2y^2+2x+3y+c=0.$$

This represents a pair of st. lines if

$$2c+2.1.\tfrac{3}{2}.\tfrac{3}{2}-1.\tfrac{9}{4}-2.1-c.\tfrac{9}{4}=0,$$

i.e., $$c=1.$$

The equation of the asymptotes is therefore

$$x^2+3xy+2y^2+2x+3y+1=0,$$

and the equation of the conjugate hyperbola is

$$x^2+3xy+2y^2+2x+3y=0.$$

Ex. 2. Find the asymptotes of

$$2x^2-xy-y^2+2x-2y+2=0,$$

and the general equation of all hyperbolas having the same asymptotes.

The equation of the asymptotes will be seen to be

$$2x^2-xy-y^2+2x-2y=0 \qquad \dots \quad (1)$$

The equation

$$2x^2-xy-y^2+2x-2y+\lambda=0,$$

where λ is a constant is the required general equation for equation (1) will differ from it by the constant λ.

Ex. 3. Find the equation to the hyperbola, whose asymptotes are the straight lines $x+2y+3=0$, and $3x+4y+5=0$, and which passes through the point $(1, -1)$.

Find also the equation to the conjugate hyperbola.

Ans. $3x^2+10xy+8y^2+14x+22y+7=0$;
$3x^2+10xy+8y^2+14x+22y+23=0$.

10·5. Properties of the conjugate diameters.

(1) *If a pair of diameters be conjugate with respect to a hyperbola, they will be conjugate with respect to its conjugate hyperbola.*

The diameters $y=mx$, $y=m'x$ are conjugate with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

if
$$mm'=\frac{b^2}{a^2} \quad \ldots \quad (1).$$

Now the equation of the conjugate hyperbola differs from that of the original hyperbola in having $-a^2$ and $-b^2$ respectively instead of a^2 and b^2.

The above diameters will therefore be conjugate with respect to the conjugate hyperbola, if

$$mm'=\frac{-b^2}{-a^2}=\frac{b^2}{a^2},$$

which is the same as (1) above.

This proves the proposition.

(2) *If a diameter meets a hyperbola in real points, it will meet the conjugate hyperbola in imaginary points; and the conjugate diameter will meet the conjugate hyperbola in real points.*

Let the equations of the hyperbola and its conjugate be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

Converting these into polar co-ordinates,

$$\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=\frac{1}{r^2} \quad \ldots \quad (1),$$

$$\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=-\frac{1}{r^2} \quad \ldots \quad (2).$$

If for a given value of θ, the left hand sides are positive, the corresponding diameter has real intersections with (1) and imaginary intersections with (2).

Now let the equation to a diameter of the hyperbola be $y=mx$, and let the conjugate diameter be

$$y=m'x.$$

Then $$mm'=\frac{b^2}{a^2}.$$

The diameter $y=mx$ meets the hyperbola in points whose abscissae are given

$$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)=1,$$

i.e., $$x^2=\frac{a^2b^2}{b^2-a^2m^2}.$$

The intersections are thus real if $m^2<\frac{b^2}{a^2}$. In this case $m'^2>\frac{b^2}{a^2}$.

The x-co-ordinates of the points where the diameter $y=m'x$ meets the conjugate hyperbola are given by

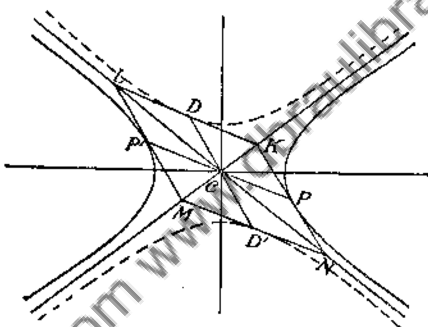
$$x^2\left(\frac{1}{a^2}-\frac{m'^2}{b^2}\right)=-1,$$

i.e.,
$$x^2 = \frac{a^2b^2}{a^2m'^2 - b^2}.$$

Since $m'^2 > \frac{b^2}{a^2}$, the intersections are real.

We also have therefore that *only one of a pair of conjugate diameters meets a hyperbola in real points.*

(3) *If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, then* $CP^2 - CD^2 = a^2 - b^2$.



Let P be the point $(a \sec \theta, b \tan \theta)$ on the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation to CP is
$$y = \frac{b \sin \theta}{a} . x.$$

Since the product of the slopes of CP and CD is $\frac{b^2}{a^2}$, the equation to CD is
$$y = \frac{b}{a \sin \theta} x.$$

This meets the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ in the points $(a \tan \theta,\ b \sec \theta)$, and $(-a \tan \theta, -b \sec \theta)$. The co-ordinates of D are therefore $(a \tan \theta,\ b \sec \theta)$.

Then $\qquad CP^2=a^2 \sec^2 \theta+b^2 \tan^2 \theta,$

and $\qquad CD^2=a^2 \tan^2 \theta+b^2 \sec^2 \theta.$

Subtracting,

$$CP^2-CD^2=a^2-b^2.$$

(4) *The parallelogram formed by the tangents at the extremities of conjugate diameters has its vertices lying on the asymptotes and is of constant area.*

The tangent at P is

$$\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta,$$

and the tangent at D to the conjugate hyperbola is

$$\frac{x}{a}\tan\theta-\frac{y}{b}\sec\theta=-1,$$

i.e., $$\frac{x}{a}\sin\theta-\frac{y}{b}=-\cos\theta.$$

The co-ordinates of K which is the point of intersection of these two tangents are given by

$$\frac{x}{a}=\frac{y}{b}=\frac{\cos\theta}{1-\sin\theta}.$$

K therefore lies on the asymptote $\frac{x}{a}=\frac{y}{b}$.

Similarly, the remaining angular points also lie on the asymptotes.

The area of the parallelogram formed by the tangents is four times the area of the parallelogram $CPKD$.

Now, $CD = \sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}$,

and the perpendicular 'p' from C upon the tangent at P is

$$\frac{\cos \theta}{\sqrt{\frac{1}{a^2} + \frac{\sin^2 \theta}{b^2}}}$$

$$= \frac{ab}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}}$$

$\therefore \quad CD \cdot p = ab.$

The area of the parallelogram $KLMN$ is therefore $4ab$.

Ex. Show that the portion of a tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact.

10·51. The equation of a hyperbola referred to any pair of conjugate diameters as axes.

The equation of the hyperbola referred to its transverse and conjugate axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If the co-ordinate axes are rotated such that they coincide with a pair of conjugate diameters, the equation has the form (§ 3·3)

$$Ax^2 + 2Hxy + By^2 = 1 \qquad \ldots \quad (1).$$

Since all chords parallel to one diameter are bisected by the other, $H = 0$. This simplifies (1) to

$$Ax^2 + By^2 = 1 \qquad \ldots \quad (2).$$

One of the two semi-conjugate diameters is real, and the other imaginary. If a' and $\sqrt{-1}\ b'$ be their lengths, the points $(a', 0)$, $(0, \sqrt{-1}\ b')$ lie on (2).

Substituting, we get $A=\frac{1}{a'^2}$, $B=-\frac{1}{b'^2}$.

Hence the required equation is

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1.$$

Ex. 1. If e, e' be the eccentricities of a hyperbola and of the conjugate hyperbola, then show that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

Ex. 2. Prove that the chord, which joins the points in which a pair of conjugate diameters meets the hyperbola and its conjugate, is parallel to one asymptote and is bisected by the other.

Ex. 3. Find the locus of the poles of normal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The equation of a normal to the hyperbola is

$$by\cot\theta+ax\cos\theta=a^2+b^2 \qquad \dots \quad (1)$$

If (h, k) be the pole, the polar

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1$$

must be the same as (1).

Comparing Coeffs.,

$$\frac{h/a^2}{a\cos\theta}=\frac{-k/b^2}{b\cot\theta}=\frac{1}{a^2+b^2}.$$

Hence, $\cos\theta=\frac{h}{a^3}(a^2+b^2)$,

$$\cot\theta=-\frac{k}{b^3}(a^2+b^2).$$

Eliminating θ,

$$1+\frac{b^6}{k^2(a^2+b^2)^2}=\frac{a^6}{h^2(a^2+b^2)^2},$$

or $$h^2k^2(a^2+b^2)^2=a^6k^2-b^6h^2.$$

Hence the required locus is

$$x^2y^2(a^2+b^2)^2=a^6y^2-b^6x^2.$$

Ex. 4. Show that the locus of the middle points of normal chords of the rectangular hyperbola

$$x^2-y^2=a^2$$

is $$(y^2-x^2)^3=4a^2x^2y^2.$$

Ex. 5. Show that if a chord PQ of a rectangular hyperbola subtends a right angle at a point O on the curve, PQ is a parallel to the normal at O.

Ex. 6. Prove that the polar of any point on an asymptote of a hyperbola with respect to the hyperbola is parallel to that asymptote.

Ex. 7. A straight line is drawn parallel to the conjugate axis of a hyperbola to meet it and the conjugate hyperbola in the points P and Q; show that the tangents at P and Q meet on the curve

$$\frac{y^4}{b^4}\left(\frac{y^2}{b^2}-\frac{x^2}{a^2}\right)=\frac{4x^2}{a^2},$$

and that the normals meet on the axis of x.

Ex. 8. A series of chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are tangents to the circle described on the straight line joining the foci of the hyperbola as diameter; show that the locus of their poles with respect to the hyperbola is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$

Ex. 9. A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola, and one of its diagonals is a chord of the hyperbola; show that the other diagonal passes through the centre.

If the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, and (x_1, y_1), (x_2, y_2), the co-ordinates of the extremities of one diagonal of the parallelogram, the equations to two adjacent sides will be

$$\frac{x}{a}-\frac{y}{b}=\frac{x_1}{a}-\frac{y_1}{b}$$

and

$$\frac{x}{a}+\frac{y}{b}=\frac{x_2}{a}+\frac{y_2}{b},$$

The co-ordinates of the point of intersection of these lines are

$$x=\frac{x_1+x_2}{2}-\frac{a}{2b}(y_1-y_2),\ y=\frac{y_1+y_2}{2}-\frac{b}{2a}(x_1-x_2).$$

The other diagonal passes through this point and the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Its equation therefore is

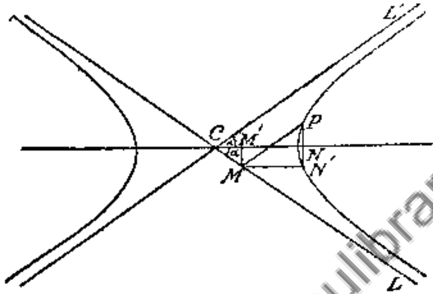
$$y-\frac{y_1+y_2}{2}=\frac{b^2(x_1-x_2)}{a^2(y_1-y_2)}\left(x-\frac{x_1+x_2}{2}\right),$$

ie., $$a^2(y_1-y_2)\,y-b^2(x_1-x_2)\,x=0,$$

the constant term vanishing since (x_1, y_1), (x_2, y_2) lie on the hyperbola. This clearly goes through the centre.

Ex. 10. Hyperbolas are drawn having a common transverse axis of length $2a$. On each is taken a point P such that its distance from the transverse axis is equal to its distance from an asymptote. Prove that the locus of P is the quartic curve $(x^2-y^2)^2=4x^2(x^2-a^2)$ referred to the common transverse axis and its perpendicular bisector as co-ordinate axes.

10.6 Equation of the hyperbola referred to its asymptotes



Let the asymptotes CL, CL' of the hyperbola whose equation referred to its axes is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \dots \quad (1),$$

be taken as the new co-ordinate axes.

If α is the angle which either asymptote makes with the transverse axis,

$$\tan\alpha=\frac{b}{a},$$

and hence $\quad \sin\alpha=\dfrac{b}{\sqrt{a^2+b^2}},\ \cos\alpha=\dfrac{a}{\sqrt{a^2+b^2}}$

Let P be any point (x, y) on the hyperbola when the equation is referred to the transverse and conjugate axes. Let a line through P be drawn parallel to CL' to meet CL in M. If the co-ordinates of P referred to CL and CL' as axes be (h, k), then

$$h=CM, \text{ and } k=MP.$$

Draw MM', perpendicular to the transverse axis and let PN produced meet the parallel to the transverse axis through M in N'.

Now $\quad x=CN=CM'+MN'=h\cos\alpha+k\cos\alpha=\dfrac{(h+k)a}{\sqrt{a^2+b^2}}$

and $\quad y=PN=PN'-MM'=k\sin\alpha-h\sin\alpha=\dfrac{(k-h)b}{\sqrt{a^2+b^2}}$.

Substituting in (1),

$$\frac{(h+k)^2}{a^2+b^2}-\frac{(k-h)^2}{a^2+b^2}=1,$$

i.e., $$hk=\frac{a^2+b^2}{4}.$$

Changing into current co-ordinates the equation to the hyperbola is

$$\mathbf{xy=\frac{a^2+b^2}{4}}$$

The equation $\mathbf{xy=c^2}$ therefore represents a hyperbola whose asymptotes are the co-ordinate axes. In the special case of the rectangular hyperbola the angle between the new axes is 90°.

The equation of the conjugate hyperbola, when referred to the asymptotes, will similarly be seen to be

$$xy=-\frac{a^2+b^2}{4}.$$

10·61. Propositions about the hyperabola $\mathbf{xy=c^2}$.

As in chapter V, or independently, the equation of the tangent at any point (x', y') of the hyperbola $xy=c^2$ is

$$\tfrac{1}{2}(xy'+yx')=c^2=x'y'.$$

This can be written as

$$\frac{x}{x'}+\frac{y}{y'}=2 \quad \dots \quad (1),$$

which is also the equation of the polar of the point (x', y').

The co-ordinates of any point on the hyperbola in terms of one variable 't' are easily seen to be $\left(ct, \frac{c}{t}\right)$.

From (1), the equation of the tangent at the point 't' is

$$\frac{x}{t}+yt=2c.$$

The equation of the chord whose middle point is (x', y') is

$$\tfrac{1}{2}(xy'+yx')-c^2=x'y'-c^2,$$

that is
$$\frac{x}{x'}+\frac{y}{y'}=2.$$

If the hyperbola be rectangular, the equation of the normal at the point (x', y') is

$$\frac{1}{x'}(y-y')-\frac{1}{y'}(x-x')=0,$$

i.e.,
$$yy'-xx'+x'^2-y'^2=0.$$

From this, the equation of the normal at the point 't' is

$$ty-t^3x+ct^4-c=0.$$

Ex. 1. The chord PP' of a hyperbola meets the asymptotes in Q, Q'. Show that $QP=P'Q'$.

Ex. 2. Prove that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of the tangent intercepted between the asymptotes is bisected at the point of contact.

The equation of the hyperbola referred to its asymptotes is $xy=c^2$. The equation of the tangent at any point (x', y') is

$$\frac{x}{x'}+\frac{y}{y'}=2.$$

This meets the axes in the points $(2x', 0)$, $(0, 2y')$. If 2α be the angle between the asymptotes, the required area of the triangle is

$$\tfrac{1}{2}\cdot 4x'y' \sin 2\alpha=4x'y' \sin\alpha \cos\alpha=\frac{4c^2ab}{a^2+b^2}=ab$$

The middle point of the portion of the tangent intercepted between the asymptotes is $\left(\frac{2x'+0}{2}, \frac{0+2y'}{2}\right)$, *i.e.*, (x', y') which is the point of contact itself.

Ex. 3. Prove that $y-mx=0$ and $y+mx=0$ are conjugate diameters of $xy=c^2$ for all values of m.

Ex. 4. Prove that the orthocentre of a triangle inscribed in a rectangular hyperbola lies on the rectangular hyperbola.

Let PQR be a triangle inscribed in the rectangular hyperbola $xy=c^2$, and let the co-ordinates of P,Q,R be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right).$$

The equation to PQ is

$$y-\frac{c}{t_1}=\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1}(x-ct_1),$$

or
$$x+yt_1t_2=c(t_1+t_2)$$

The equation of the perpendicular from R on PQ is

$$y-\frac{c}{t_3}-t_1t_2(x-ct_3)=0,$$

i.e.,
$$y+ct_1t_2t_3=t_1t_2\left(x+\frac{c}{t_1t_2t_3}\right) \quad \dots \quad (1)$$

Similarly, the equation of the perpendicular from P on QR is

$$y+ct_1t_2t_3=t_2t_3\left(x+\frac{c}{t_1t_2t_3}\right). \quad \dots \quad (2)$$

The orthocentre of the triangle, which is the common point of (1) and (2) is

$$\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right).$$

This obviously lies on the hyperbola.

Ex. 5 A circle cuts the rectangular hyperbola $xy=c^2$ in the points 't_1', 't_2', 't_3', 't_4'. Show that $t_1t_2t_3t_4=1$.

Let the circle be

$$x^2+y^2+2gx+2fy+k=0.$$

The point $\left(ct, \dfrac{c}{t}\right)$ lies on the circle, if

$$c^2t^2+\frac{c^2}{t^2}+2gct+2f\frac{c}{t}+k=0,$$

i.e., if $$c^2t^4+2gct^3+kt^2+2fct+c^2=0$$

If t_1, t_2, t_3, t_4 be the roots of this equation, we evidently have $$t_1t_2t_3t_4=1.$$

Ex. 6. A circle with fixed centre $(3h, 3k)$ and of variable radius cuts the rectangular hyperbola $x^2-y^2=9a^2$ at the points A, B, C, D; prove that the locus of the centroid of the triangle ABC is given by $(x-2h)^2-(y-2k)^2=a^2$.

[Math. Tripos 1941]

EXAMPLES ON CHAPTER X

1. Find the asymptotes of the hyperbola $xy-2x-3y=0$, and the equation of the conjugate hyperbola.

2. A circle cuts two fixed perpendicular lines so that each intercept is of given length. Prove that the locus of the centre of the circle is a rectangular hyperbola.

3. If two sides of a triangle are given in position and the perimeter given in magnitude, prove that the middle point of the third side describes a hyperbola.

4. From a point P of a rectangular hyperbola perpendiculars PM, PM' are drawn to the asymptotes. Show that $PM.PM'$ is constant.

The chord QQ' of a rectangular hyperbola is parallel to the tangent at P and $QM, Q'M'$, PN are drawn perpendicular to either asymptote. Show that $QM.Q'M' = PN^2$.

5. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point t meets the curve again at the point t' such that $t^3t' = -1$.

6. Show that the locus of the pole of any straight line with respect to a coaxal system of circles is a hyperbola one of whose asymptotes is perpendicular to the given line, and the other is parallel to the radical axis of the system.

7. P, Q are two variable points on the rectangular hyperbola $xy = c^2$, such that the tangent at Q passes through the foot of the ordinate of P. Show that the locus of the intersection of tangents at P and Q is a hyperbola with the same asymptotes as the given hyperbola.

8. Show that the co-ordinates of the point of intersection of two tangents to a hyperbola referred to its asymptotes as axes are harmonic means between the co-ordinates of the points of contact.

9. Find the locus of a point such that the angle between the tangents from it to a hyperbola is equal to the angle between the asymptotes of the hyperbola.

10. Prove that the four equations

$$b(x \pm \sqrt{x^2 - a^2}) = a(y \pm \sqrt{y^2 + b^2})$$

represent respectively the portions of a hyperbola referred to its axes which lie in the four quadrants.

11. The straight line $ax + by = 1$ meets the hyperbola $xy = c^2$ in P and Q. Show that the lines CP, CQ (C being the centre of the hyperbola) are perpendicular if

$$c^2e^2(a^2 + b^2) - (2 - e^2)(2c^2ab - 1) = 0,$$

where e is the eccentricity of the hyperbola.

12. If the normals at (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) on the rectangular hyperbola $xy=c^2$ meet in the point (α, β), prove that

$$\alpha = x_1+x_2+x_3+x_4,$$

$$\beta = y_1+y_2+y_3+y_4,$$

and
$$x_1x_2x_3x_4 = y_1y_2y_3y_4 = -c^4.$$

13. A circle cuts the rectangular hyperbola $xy=1$ in the points (x_r, y_r), $r=1, 2, 3, 4$. Prove that

$$x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1.$$

14. A rectangular hyperbola and a circle intersect in four points. Show that the centre of mean position of the points bisects the distance between the centres of the two curves.

15. A rectangular hyperbola whose centre is C is cut by any circle of radius r in the four points P, Q, R, S; prove that

$$CP^2+CQ^2+CR^2+CS^2=4r^2.$$

16. Prove that if the normals at P,Q,R,S on a rectangular hyperbola intersect in a point, then the circle PQR passes through the other extremity of the diameter through S.

17. Normals are drawn to a rectangular hyperbola at the ends of a chord whose direction is given. Prove that the locus of their intersection is another rectangular hyperbola, whose asymptotes make with the asymptotes of the given hyperbola angles equal and opposite to those made by the given direction.

18. If the hyperbola be rectangular and its equation be $xy=c^2$, prove that the locus of the middle points of chords of constant length $2d$ is

$$(x^2+y^2)(xy-c^2)=d^2xy.$$

19. Prove that in any rectangular hyperbola the rectangle under the distances of any point of the curve

from two fixed tangents is to the square on the distance from their chord of contact as $\cos\varphi : 1$, where φ is the angle between the tangents.

20. From two points (x_1, y_1), (x_2, y_2) are drawn tangents to the rectangular hyperbola $xy=c^2$; prove that the conic passing through the two points and through the four points of contact will be a circle if

$$x_1y_2+x_2y_1=4c^2, \text{ and } x_1x_2=y_1y_2.$$

21. Show that an infinite number of triangles can be inscribed in the rectangular hyperbola $xy-c^2=0$ whose sides all touch the parabola $y^2-4ax=0$.

22. With the point (α, β) as centre a family of circles is drawn to cut the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Prove that the locus of the middle points of the chords of intersection is a rectangular hyperbola, which passes through the centre of the given conic.

23. Three tangents are drawn to the rectangular hyperbola $xy=a^2$ at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and form a triangle whose circumcircle passes through the centre of the hyperbola. Prove that

$$\frac{\Sigma x_1}{x_1x_2x_3}+\frac{\Sigma y_1}{y_1y_2y_3}=0,$$

and that the centre of the circle lies on the hyperbola.

24. If O is the centre of the rectangular hyperbola through the four points A, B, C, D whose co-ordinates are $\left(t, \frac{1}{t}\right)$ where $t=a, b, c, d$ and if the perpendiculars from O to BC, AD meet AD, BC respectively in P, P', prove that if $abcd+1\neq 0$, the equation of the circle on PP' as diameter is

$$(x+bcy-b-c)(x+ady-a-d)+(bcx-y)(adx-y)=0.$$

[Math. Tripos 1942].

25. An ellipse and a hyperbola are so related that the asymptotes of the hyperbola are conjugate diameters of the ellipse; prove that by a proper choice of axes their equations may be expressed in the forms

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=m.$$

26. Triangles are inscribed in the circle $x^2+y^2=2ax$, and their sides touch the hyperbola $x^2-y^2=a^2$. Prove that the locus of the orthocentre is the circle

$$x^2+y^2+2ax=0$$

27. A circle is described on a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameter which is parallel to the straight line $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=0$. Prove that the locus of the pole with respect to the circle of the straight line joining the two other common points is the hyperbola

$$\frac{a^2x^2}{\cos^2\alpha}-\frac{b^2y^2}{\sin^2\alpha}=a^4-b^4.$$

CHAPTER XI

THE TRACING OF THE CONICS

11·1. Representation of a conic by the general equation of the second degree. In Chapter V (§ 5·2) we have seen that the equation to a conic is of the second degree. We shall now show that every Cartesian equation of the second degree represents a conic.

Let us consider the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad (1),$$

and suppose that the axes are *rectangular*.

If the axes are turned through an angle θ given by $\tan 2\theta=\frac{2h}{a-b}$, the xy term of the new equation vanishes (§ 3·6).

Let the transformed equation be

$$a'X^2+b'Y^2+2g'X+2f'Y+c=0 \quad \ldots \quad (2),$$

which shows that the expression $ax^2+2hxy+by^2$ has changed into $a'X^2+b'Y^2$.

From the theory of invariants,

$$a+b=a'+b',$$

and

$$ab-h^2=a'b'.$$

Two cases now arise.

(1) $ab-h^2=0$. In this case either $a'=0$ or $b'=0$. If $a'=0$, equation (2) reduces to

$$b'Y^2+2g'X+2f'Y+c=0,$$

which is a parabola, of which the axis is parallel to the x-axis.

If $b'=0$, equation (2) will represent a parabola of which the axis is parallel to the y axis.

(2) $ab-h^2 \neq 0$. In this case both a' and b' are different from zero. Equation (2) can now be written as

$$a'\left(X^2+\frac{2g'}{a'}X\right)+b'\left(Y^2+\frac{2f'}{b'}Y\right)+c=0,$$

or $$a'\left(X+\frac{g'}{a'}\right)^2+b'\left(Y+\frac{f'}{b'}\right)^2=\frac{g'^2}{a'}+\frac{f'^2}{b'}-c,$$

and therefore represents an ellipse or a hyperbola with centre at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ according as the signs of a' and b' are the same or different. If the signs are different and in addition $a'+b'=0$, the hyperbola will be rectangular.

But a' and b' have the same sign if $a'b'$ is positive, that is, if $ab-h^2$ is positive, and opposite signs if $a'b'$ is negative, that is, if $ab-h^2$ is negative.

Equation (1) therefore represents an *ellipse* if $ab-h^2$ is *positive* and a *hyperbola* if $ab-h^2$ is *negative*. In the special case when $a+b=0$, the hyperbola is rectangular.

11·11. Oblique axes. In the preceding article the co-ordinate axes were assumed to be rectangular. If the axes are inclined at an angle ω, we transform to rectangular axes with the same origin and suppose that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots \quad (1)$$

becomes $$a'X^2+2h'XY+b'Y^2+2g'X+2f'Y+c=0.$$

From the preceding article, this represents

(*i*) a parabola if $a'b'-h'^2=0$,

(*ii*) an ellipse if $a'b'-h'^2>0$,

(*iii*) a hyperbola if $a'b'-h'^2<0$,

and (*iv*) a rectangular hyperbola if $a'+b'=0$.

By invariants (§ 3·7),

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}}=a'+b',$$

$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\pi/2}=a'b'-h'^2.$$

Equation (1) thus represents a parabola if $ab-h^2=0$, an ellipse if $ab-h^2>0$, a hyperbola if $ab-h^2<0$, and a rectangular hyperbola if $a+b-2h\cos\omega=0$.

11·12. Summary. We shall summarise below the results in connection with the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

(*i*) A *pair of straight lines* if

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

(*ii*) A *pair of parallel straight lines* if

$$\Delta=0, \text{ and } h^2=ab.$$

(*iii*) A *circle* if

$$a=b, \text{ and } h=0.$$

If the axes are oblique, the condition is $a:b:h=1:1:\cos\omega$.

(*iv*) A *parabola* if

$$h^2=ab, \text{ and } \Delta\neq 0,$$

i. e., if the second degree terms form a perfect square and the condition for two straight lines is not satisfied.

(*v*) An *ellipse* if

$ab-h^2>0$, and the condition for a circle is not satisfied.

(*vi*) A *hyperbola* if

$ab-h^2<0$, and $\triangle \neq 0$.

(*vii*) A *rectangular hyperbola* if

$a+b=0$, and $\triangle \neq 0$.

If the axes are oblique, the condition is

$a+b-2h \cos \omega=0$, $\triangle \neq 0$.

Note—The axes will be rectangular unless otherwise stated.

11·2. Axis and latus rectum of the parabola represented by the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad (1)$$

We have seen that this equation represents a parabola if the second degree terms form a perfect square. For the sake of convenience, therefore, let $a=\alpha^2$, $b=\beta^2$ and $h=\alpha\beta$.

Equation (1) then becomes

$$(\alpha x+\beta y)^2+2gx+2fy+c=0 \quad \ldots \quad (2)$$

Now in a parabola the square of the distance of any point on the curve from the axis is equal to its distance from the tangent at the vertex multiplied by the length of the latus rectum.

Further, since the axis and the tangent at the vertex are perpendicular, we shall write (2) in a form which expresses the above property in relation to two perpendicular straight lines.

Now equation (2) can be written as

$$(\alpha x+\beta y+\lambda)^2=2x(\alpha\lambda-g)+2y(\beta\lambda-f)+\lambda^2-c$$

We shall choose λ such that the lines

$$\alpha x+\beta y+\lambda=0 \qquad \ldots \quad (3)$$

and $$2x(\alpha\lambda-g)+2y(\beta\lambda-f)+\lambda^2-c=0 \qquad \ldots \quad (4)$$

are perpendicular.

The condition for this is

$$(\alpha\lambda-g)\alpha+(\beta\lambda-f)\beta=0,$$

which gives

$$\lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

With this value of λ equation (3) represents the axis, equation (4) the tengent at the vertex and the point of intersection of (3) and (4) is the vertex of the parabola.

To find the latus rectum, let the equation be written as

$$\left(\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}\right)^2(\alpha^2+\beta^2)$$

$$=2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}.$$

$$\left\{\frac{(\alpha\lambda-g)x+(\beta\lambda-f)y+\frac{1}{2}(\lambda^2-c)}{\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}\right\}$$

The length of the latus rectum is thus

$$\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{(\alpha^2+\beta^2)^2}$$

Substituting for λ, the expression under the radical sign becomes

$$\frac{\beta^2(\alpha f-\beta g)^2+\alpha^2(\beta g-\alpha f)^2}{(\alpha^2+\beta^2)^2},$$

i.e., $$\frac{(\alpha f-\beta g)^2}{\alpha^2+\beta^2}$$

The length of the latus rectum is, therefore, the numerical value of

$$\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}.$$

Ex. 1. Trace the parabola

$$9x^2+24xy+16y^2-2x+14y+1=0,$$

and find the co-ordinates of its focus.

The equation can be written as

$$(3x+4y+\lambda)^2=2x(1+3\lambda)+2y(4\lambda-7)+\lambda^2-1.$$

The lines $3x+4y+\lambda=0$ and $2x(1+3\lambda)+2y(4\lambda-7)+\lambda^2-1=0$ are perpendicular if

$$-\tfrac{3}{4}.\left(-\frac{1+3\lambda}{4\lambda-7}\right)=-1,$$

from which $\lambda=1.$

The equation now becomes

$$(3x+4y+1)^2=8x-6y,$$

which we write as

$$\left(\frac{3x+4y+1}{5}\right)^2 25=10.\frac{8x-6y}{10}.$$

The equation of the axis is

$$3x+4y+1=0.$$

The equation of the tangent at the vertex is

$$8x-6y=0$$

and the latus rectum is $\frac{2}{5}$.

The co-ordinates of the vertex are $\left(-\frac{3}{25}, -\frac{4}{25}\right).$

The parabola meets the x-axis in points given by

$$9x^2-2x+1=0,$$

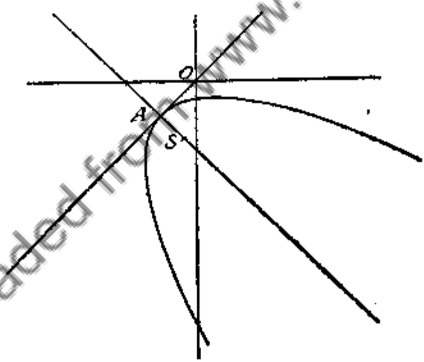
of which the roots are imaginary.

The intersections with the y-axis are given by

$$16y^2+14y+1=0,$$

i.e.,
$$y=\frac{-14\pm\sqrt{132}}{32}$$

The parabola therefore has imaginary intersections with the x-axis and meets the y-axis in the points $\left(0, \frac{-7\pm\sqrt{33}}{16}\right)$. From this it will be obivous that the parabola lies to the right of the tangent at the vertex as shown in the figure.



The focus is a point on the axis distant $\frac{1}{10}$ from the vertex A. Its co-ordinates are therefore

$$\left(-\frac{3}{25}+\frac{1}{10}\cos\alpha,\ -\frac{4}{25}-\frac{1}{10}\sin\alpha\right)$$

where $\tan\alpha=\frac{3}{4}$.

Thus the focus is the point $(-\frac{1}{25}, -\frac{11}{50})$.

Note :— The student is advised always to find out the intersections with the co-ordinate axes while tracing a parabola.

Ex. 2. Trace the parabola

$$x^2-2xy+y^2-3x+y-2=0$$

and determine the equation of its axis and the co-ordinates of its focus. *Ans.* $x-y=1, (-\frac{7}{8}, -\frac{15}{8})$

11·3. Eccentricity of the Central Conic

$$\mathbf{ax^2+2hxy+by^2=1.}$$

We know that the equation $ax^2+2hxy+by^2=1$ represents a conic of which the centre is at the origin (§5·81 Cor.). Let the axes of co-ordinates be so rotated as to coincide with the principal axes of the conic. If the transformed equation is $\alpha x^2+\beta y^2=1$, we have, by invariants,

$$\alpha+\beta=a+b,$$

and
$$\alpha\beta=ab-h^2.$$

Also, if e be the eccentricity, and $\alpha<\beta$, we have

$$e^2=\frac{\beta-\alpha}{\beta},$$

$$\therefore \quad \frac{e^2}{2-e^2}=\frac{\beta-\alpha}{\beta+\alpha}=\frac{\sqrt{(a+b)^2-4(ab-h^2)}}{a+b}$$
$$=\frac{\sqrt{(a-b)^2+4h^2}}{a+b}.$$

Squaring and simplifying,

$$e^4(ab-h^2)+\{(a-b)^2+4h^2\}(e^2-1)=0,$$

which gives the value of e.

11·4. Asymptotes of the general conic. We know that the equation of the asymptotes differs from that of the conic only by a constant.

The asymptotes of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

will be given by

$$ax^2+2hxy+by^2+2gx+2fy+c+c'=0 \qquad (1),$$

where c' is so chosen that equation (1) represents a pair of straight lines.

The condition for this is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+c' \end{vmatrix}=0,$$

from which

$$c'(ab-h^2)+\Delta=0,$$

where

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Hence the equation of the asymptotes is

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0.$$

Ex. 1. If $(\bar{x}, \bar{y})$ be the centre of the hyperbola

$$f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

prove that the equation of the asymptotes is

$$f(x, y)=f(\bar{x}, \bar{y}).$$

Since $(\bar{x}, \bar{y})$ is the centre,

$$a\bar{x}+h\bar{y}+g=0$$

$$h\bar{x}+b\bar{y}+f=0.$$

Multiplying the first by x, second by y and adding,

$$ax^2+2hxy+by^2+gx+fy=0.$$

So that,

$$f(x, y)-gx+fy+c=k \text{ say}.$$

We then have

$$ax+hy+g=0$$

$$hx+by+f=0$$

$$gx+fy+c-k=0.$$

Eliminating $\bar{x}$, $\bar{y}$,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-k \end{vmatrix}=0$$

i.e., $$\triangle=k(ab-h^2),$$

or $$\frac{\triangle}{ab-h^2}=k=f(\bar{x}, \bar{y}).$$

Hence from the above article the equation to the asymptotes is

$$f(x, y)=f(\bar{x}, \bar{y}).$$

Ex. 2. Show that the equation of the hyperbola conjugate to

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is $$ax^2+2hxy+by^2+2gx+2fy+c-\frac{2\triangle}{ab-h^2}=0.$$

Ex. 3. If the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ be a rectangular hyperbola, show that its equation referred to its asymptotes will be

$$2(h^2-ab)^{3/2}xy-k=0.$$

Determine the constant k.

We know from §5·81 that if the origin is transferred to the centre, the axes retaining their directions the given equation will become

$$ax^2+2hxy+by^2+\frac{\triangle}{ab-h^2}=0.$$

If the axes are now rotated so as to coincide with the asymptotes, the above equation will transform into

$$2\lambda xy+\frac{\triangle}{ab-h^2}=0.$$

By invariants,

$$h^2-ab=\lambda^2, \ i.e., \quad \lambda=\pm\sqrt{h^2-ab}.$$

The equation of the conic, referred to the asymptotes, therefore is

$$2(h^2-ab)^{3/2}xy\pm\triangle=0,$$

the plus sign being taken when the hyperbola lies in the second and fourth quadrants and the minus sign when the hyperbola lies in the first and third quadrants, provided $\triangle$ is $+ve$. If $\triangle$ is $-ve$, the above order of signs is reversed.

The value of k is evidently $\pm\triangle$.

11·5. The lengths and position of the axes of the conic $ax^2+2hxy+by^2=1$.

Let us consider the circle

$$x^2+y^2=r^2 \qquad \dots \quad (1),$$

of which the centre is at the centre of the conic

$$ax^2+2hxy+by^2=1 \qquad \dots \quad (2).$$

The equation to the pair of straight lines joining the origin to the intersections of (1) and (2) is

$$ax^2+2hxy+by^2=\frac{x^2+y^2}{r^2},$$

or $$\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0 \quad \ldots \quad (3).$$

These lines will coincide in case they lie along either axis of the conic, for the circle and the conic will then touch each other.

The length of a semi-axis of the conic is, therefore, that value of r for which (3) represents two coincident straight lines.

From this,

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2.$$

If the roots of this quadratic in r^2 be r_1^2 and r_2^2, and if both r_1^2 and r_2^2 be positive, which will be the case when the conic is an ellipse, the lengths of the semi-axes will be r_1 and r_2 the greater value representing the semi-major axis. If the conic be a hyperbola one root of the above quadratic will be positive, and the other negative. If r_1^2 is the positive root the length of the semi-transverse axis is r_1. The length of the semi-conjugate axis corresponding to the negative root r_2^2 is r_2.

When the left hand member of equation (3) is a perfect square, we can write it as

$$\left[\left(a-\frac{1}{r^2}\right)x+hy\right]^2=0.$$

Substituting for r^2, the equations of the axes are

$$\left(a-\frac{1}{r_1^2}\right)x+hy=0,$$

and $$\left(a-\frac{1}{r_2^2}\right)x+hy=0.$$

In the case of the ellipse, the equation of the major axis is the one corresponding to the greater of the values r_1^2, r_2^2.

In the case of the hyperbola the equation of the transverse axis corresponds to the positive root of the quadratic in r^2.

If only the length of the axes is to be determined, we can also use the method of § 11·3 and evalute α and β.

Aliiter. We can obtain the direction and magnitude of the axes of the central conic $ax^2+2hxy+by^2=1$ alternatively thus :

If the co-ordinate axes are rotated through an angle θ

given by $$\tan 2\theta = \frac{2h}{a-b} \quad \dots \quad (1),$$

the xy term in the equation of the conic disappears. The new axes, therefore, coincide with the principal axes of the conic.

From equation (1),

$$\frac{2 \tan \theta}{1-\tan^2 \theta} = \frac{2h}{a-b} = \frac{1}{k} \text{ say,}$$

$$\therefore \quad \tan^2\theta + 2k \tan \theta - 1 = 0 \quad \dots \quad (2).$$

Let θ_1 and θ_2 be the two values of θ which satisfy (2).

Evidently, $\tan \theta_1 \cdot \tan \theta_2 = -1$.

Therefore $$\theta - \theta_2 = \frac{\pi}{2}.$$

The equation of the conic in polar co-ordinates is

$$r^2(a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) = 1 = \cos^2 \theta + \sin^2 \theta.$$

$$\therefore \quad r^2 = \frac{1+\tan^2 \theta}{a+2h \tan \theta + b \tan^2 \theta} \quad \dots \quad (3).$$

Substituting in (3) the values of $\tan \theta$ obtained from (2), we get the lengths of the axes.

The equations of the axes are

$$y = x \tan \theta_1,$$
$$y = x \tan \theta_2.$$

11·6. Co-ordinates of the foci.

If the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$

be an ellipse, the lengths of the semi-major and semi-minor axes being r_1, r_2, the eccentricity e of the conic is

$$\sqrt{1-\frac{r_2^2}{r_1^2}}.$$

The foci are points on the major axis at a distance er_1 from the centre, (x', y') say, of the conic.

If θ is the inclination of the major axis to the x-axis, the co-ordinates of the foci are

$$(x' \pm er_1 \cos\theta,\ y' \pm er_1 \sin\theta).$$

The same holds for the hyperbola provided r_1 is its semi-transverse axis.

Ex. 1. If the co-ordinate axes be inclined at an angle ω, show that the principal axes of lengths $2r_1$, $2r_2$ of the conic $ax^2+2hxy+by^2=1$ are

$$\left(a-\frac{1}{r_1^2}\right)x+\left(h-\frac{\cos\omega}{r_1^2}\right)y=0,$$

and $$\left(a-\frac{1}{r_2^2}\right)x+\left(h-\frac{\cos\omega}{r_2^2}\right)y=0.$$

Hint. The equation of the circle is

$$\frac{x^2+2xy\cos\omega+y^2}{r^2}=1.$$

The result is obtained on proceeding as in the first method of § 11·5.

Ex. 2. Trace the conic

$$17x^2+12xy+8y^2-46x-28y+33=0,$$

and find its axes and the co-ordinates of its foci.

Since $17.8-36$ is positive the conic is an ellipse.

The equations giving the co-ordinates of the centre are (§ 5 8)

$$17x+6y-23=0$$

$$6x+8y-14=0.$$

Solving, the centre is the point (1, 1).

The equation of the ellipse referred to parallel axes through (1, 1) is (§ 5·81)

$$17x^2+12xy+8y^2-23.1-14.1+33=0,$$

i.e., $$17x^2+12xy+8y^2=4 \qquad \dots (1).$$

The equation of the pair of straight lines joining the origin to the intersections of (1) and the circle $x^2+y^2=r^2$ is

$$17x^2+12xy+8y^2=\frac{4(x^2+y^2)}{r^2},$$

or $$x^2\left(17-\frac{4}{r^2}\right)+12xy+y^2\left(8-\frac{4}{r^2}\right)=0 \qquad \dots (2)$$

If r is a semi-axis,

$$\left(17-\frac{4}{r^2}\right)\left(8-\frac{4}{r^2}\right)=36,$$

i.e., $$100r^4-100r^2+16=0,$$

whence $$r^2=\tfrac{4}{5} \text{ or } \tfrac{1}{5}.$$

The equations of the major and minor axes from (2), are

$$2x+y=0, \; x-2y=0$$

Referred to the old origin, the equation of the major axis is $2(x-1)+y-1=0$, *i.e.*, $2x+y-3=0$.

The equation of the minor axis referred to the old origin is $$x-2y+1=0.$$

To trace the ellipse we shall also find out the intersections of the ellipse with the co-ordinate axes.

The distances from the origin of the points in which the ellipse cuts the x-axis are the roots of the equation

$$17x^2-46x+33=0.$$

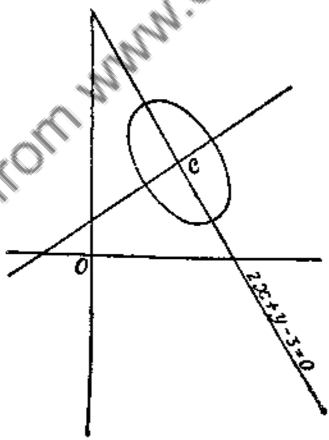
Since $46^2-4.17.33$ is negative, the roots are imaginary.

The distances from the origin of the points in which the ellipse cuts the y-axis are given by

$$8y^2-28y+33=0,$$

the roots of which are also imaginary.

The ellipse therefore does not cut the co-ordinate axes. The figure is as shown below.



The eccentricity of the ellipse is

$$\sqrt{1-\tfrac{1}{4}}=\frac{\sqrt{3}}{2}$$

The co-ordinates of the foci are therefore

$$\left\{\left(1 \pm \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}\right), \left(1 \pm \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}}\right)\right\}$$

i.e., $$\left(1 \pm \frac{\sqrt{3}}{5}, 1 \pm \frac{2\sqrt{3}}{5}\right).$$

Ex. 3. Trace the conic

$$x^2 + 4xy + y^2 - 2x + 2y + 4 = 0.$$

Since $1.1 - 4$ is negative, the conic is a hyperbola.

The centre is the point given by

$$x + 2y - 1 = 0,$$

and $$2x + y + 1 = 0.$$

Solving, the co-ordinates of the centre are $(-1, 1)$.

The equation of the hyperbola referred to parallel axes through the centre is

$$x^2 + 4xy + y^2 - 1.(-1) + 1 \cdot 1 + 4 = 0,$$

or $$x^2 + 4xy + y^2 = -6 \qquad \dots \quad (1)$$

The lines joining the centre with the intersection of (1) and the circle $x^2 + y^2 = r^2$ are

$$x^2 + 4xy + y^2 = -\frac{6(x^2 + y^2)}{r^2},$$

or $$x^2\left(1 + \frac{6}{r^2}\right) + 4xy + y^2\left(1 + \frac{6}{r^2}\right) = 0.$$

If r is a semi-axis,

$$\left(1 + \frac{6}{r^2}\right)^2 = 4,$$

i. e., $$1 + \frac{6}{r^2} = \pm 2,$$

whence $r^2 = 6$ or -2.

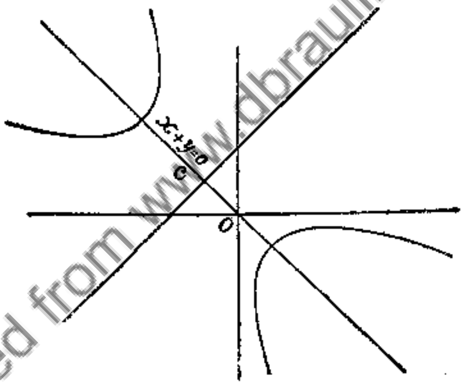
The semi-transverse axis of the hyperbola is thus of length $\sqrt{6}$ and its equation is $x+y=0$.

The hyperbola cuts the x—axis in points whose co-ordinates are $(x_1, 0)$, $(x_2, 0)$ where x_1, x_2 are the roots of the equation

$$x^2-2x+4=0.$$

From this we see that both x_1 and x_2 are imaginary.

The intersections with y—axis will similarly be seen to be imaginary.



The hyperbola is as traced above.

EXAMPLES ON CHAPTER XI

1. Trace the following conics :

(i) $x^2+2xy+y^2-2x-1=0$

(ii) $16x^2-24xy+9y^2-104x-172y+44=0$

(iii) $8x^2+4xy+5y^2=24(x+y)$

(iv) $x^2-5xy+y^2+8x-20y+15=0$

(v) $x^2-3xy+y^2+10x-10y+21=0$

(vi) $11x^2+4xy+14y^2-26x-32y+23=0$

(vii) $22x^2-12xy+17y^2-112x+92y+178=0$

(viii) $144x^2-120xy+25y^2+100x-98y+73=0$

(ix) $36x^2+24xy+29y^2-72x+126y+81=0$

(x) $9x^2-24xy+16y^2-18x-101y+19=0$

(xi) $x^2+4xy+4y^2+3x+6y+2=0$

2. Show that the conic

$$x^2+y^2-4xy-2x-20y-11=0$$

is a hyperbola. Find the coordinates of the centre, and show that the distance between the vertices of the two branches of the hyperbola is 12.

[Math. Tripos, 1947]

3. Prove that the equation

$$9x^2-6xy+y^2-13x+y+8=0$$

represents a parabola.

Determine the length of the semi-latus rectum of this parabola, the coordinates of the vertex and the equations of the axis and directrix.

[Birmingham, 1944]

4. Trace the conic

$$17x^2-12xy+8y^2+46x-28y+17=0$$

5. Show that the centre of the conic

$$x^2+24xy-6y^2+28x+36y+16=0$$

lies at the point $(-2, -1)$.

Find the equations of the axes of the conic and of its asymptotes and the lengths of its axes. [Wales, 1945]

6. Trace the conic

$$32x^2+52xy-7y^2-64x-52y-148=0,$$

and find the co-ordinates of its foci.

7. Trace the conic

$$8x^2-24xy+15y^2+48x-48y=0.$$

Find its eccentricity.

8. Find the position of the centre and the lengths of the axes of the conic

$$97x^2-60xy+72y^2-314x+348y+37=0.$$

Sketch the conic. [Edinburgh, 1946

9. Show that the latus rectum of the parabola

$$(a^2+b^2)(x^2+y^2)=(bx+ay-ab)^2$$

is $$2ab\div(a^2+b^2)^{\frac{1}{2}}$$

10. Show that the curve given by the equations

$$x=at^2+bt+c, \quad y=a't^2+b't+c'$$

is a parabola of latus rectum

$$\frac{(a'b-ab')^2}{(a^2+a'^2)^{3/2}}$$

11. Show that the semi-axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are given by

$$(ab-h^2)^3r^4+\triangle(a+b)(ab-h^2)r^2+\triangle^2=0,$$

where $$\triangle\equiv abc+2fgh-af^2-bg^2-ch^2$$

12. Trace carefully the conic

$$x^2-4xy-2y^2+10x+4y=0,$$

finding its centre, axes and asymptotes.

13. Draw a rough sketch of the conic

$$3(x^2+y^2)+2xy=4\sqrt{2}(x+y).$$

Determine the foci, and show that the origin lies at an extremity of one of its principal axes.

14. Prove that the principal axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are parallel to the lines

$$h(x^2-y^2)=(a-b)xy.$$

Find the equations of the principal axes of

$$2x^2+12xy-7y^2-16x+2y-3=0.$$

[Math. Tripos, 1940]

[**Hint**. The axes bisect the angles between the asymptotes, which are parallel to the lines $ax^2+2hxy+by^2=0$]

15. Prove that if the centre of the conic whose equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

be (α, β), then the equation of the asymptotes can be written in the form

$$ax^2+2hxy+by^2+2gx+2fy-g\alpha-f\beta=0.$$

Two fixed points A and B have co-ordinates (a, b) and (a', b') respectively. Two variable points P and Q are taken on the x-axis, with PQ constant and equal to k (Q being on the right of P). The straight lines AP and BQ meet at T. Prove that the locus of T is a hyperbola and find the co-ordinates of its centre. Also prove that one of its asymptotes is the x-axis, and find the equation of the other asymptote.

16. Show that the lengths of the semi-axes of the conic

$$ax^2+2hxy+by^2=1$$

are the roots of the equation

$$(ab-h^2)r^4-(a+b)r^2+1=0.$$

Trace the conic

$$108x^2-312xy+17y^2+504x+522y-387=0,$$

showing the axes, centre, and lengths of the semi-axes.

[Indian Audit & Accts. Service, 1940]

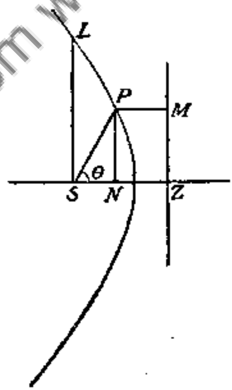
CHAPTER XII

POLAR EQUATION OF A CONIC

12·1. Introductory Remarks. The polar equation of a conic section can always be derived from its Cartesian equation referred to rectangular axes by writing $x=r\cos\theta$, $y=r\sin\theta$. It is however advantageously used when the pole is at a focus of the conic.

In this chapter we shall therefore be concerned with polar equations of conics when the pole is at a focus.

12·2. Polar equation of a conic, its focus being the pole.



Let P be any point (r, θ) on a conic of which the focus S is the pole and the perpendicular SZ from S on the directrix ZM is the initial line.

If e be the eccentricity of the conic, then

$$SP = e.PM \qquad \dots \quad (1),$$

where PM is the perpendicular on the directrix.

But $PM = NZ = SZ - SN = \frac{l}{e} - r \cos \theta$, l being the semi-latus rectum SL of the conic.

Hence, from (1),

$$r = e\left(\frac{l}{e} - r \cos \theta\right),$$

i.e.,

$$\mathbf{\frac{l}{r} = 1 + e \cos \theta},$$

which is the required polar equation.

If the positive direction of the initial line were ZS instead of SZ, the equation of the conic would have been obtained as

$$\frac{l}{r} = 1 - e \cos \theta.$$

If the axis SZ of the conic is inclined at an angle α to the initial line, the equation of the conic will be

$$\frac{l}{r} = 1 + e \cos (\theta - \alpha),$$

for the angle between SP and SZ in this case is $\theta - \alpha$.

12·21. Equations of the directrices. If (r, θ) be the co-ordinates of any point on the directrix corresponding to the focus S, then

$$r \cos \theta = SZ = \frac{l}{e}$$

The equation of the directrix corresponding to the focus which has been chosen as the pole is, therefore,

$$\frac{l}{r} = e \cos \theta.$$

The distance of the other directrix from S is

$$\frac{l}{e}+2CZ,$$

where C is the centre of the conic.

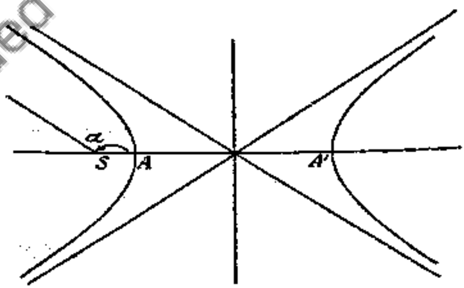
But $CZ=\frac{a}{e}=\frac{l}{e(e^2-1)}$.

Hence the equation of the other directrix is

$$r\cos\theta=-\left(\frac{l}{e}+\frac{2l}{e(e^2-1)}\right),$$

i.e., $$\frac{l}{r}=-\frac{e(e^2-1)}{e^2+1}\cos\theta.$$

12·22. Tracing the conic $\frac{l}{r}=1+e\cos\theta$. The student should find no difficulty in tracing the conic when $e=0$, 1 or <1. When $e>1$, that is, when the conic is a hyperbola, the equation $\frac{l}{r}=1+e\cos\theta$ represents only points on the branch nearer the pole if the radius vector is positive. Points on the further branch are obtained by taking negative values of the radius vector.



As θ increases from 0 to $\frac{\pi}{2}$, r increases from $\frac{l}{1+e}$ to l. As θ increases still further, $\cos\theta$ becomes negative and r goes on increasing. For $\cos\theta = -\frac{1}{e}$, r becomes infinite. A further increase in the value of θ, howsoever small, will make r negative. Hereafter r continues to be negative until $\theta = \pi$, when $r = -\frac{l}{e-1}$, which corresponds to A' on the further branch. If θ increases beyond π, r remains negative until θ approaches $\left\{2\pi - \cos^{-1}\left(-\frac{1}{e}\right)\right\}$ when $r \to -\infty$. So that for values of θ between α and $2\pi - \alpha$, $\alpha = \cos^{-1}\left(-\frac{1}{e}\right)$, the branch of the hyperbola farther from the pole is traced. For θ increasing from $2\pi - \alpha$ to 2π, r maintains a positive value, and points on the lower half of the branch nearer the pole are obtained.

The points at infinity on the curve are given by

$$\cos\theta = -\frac{1}{e}.$$

12·3. Asymptotes. From the preceding article it is apparent that the asymptotes of the conic

$$\frac{l}{r} = 1 + e\cos\theta,$$

are the two lines through the centre in the directions given by

$$\cos\theta = -\frac{1}{e}.$$

The co-ordinates of the centre are $(ae, 0)$, where a is the semi-transverse axis of the hyperbola.

The length of the perpendicular from S upon an asymptote is $ae\sin\alpha = a\sqrt{e^2-1}$, $\cos\alpha$ being $-\frac{1}{e}$.

The perpendicular makes an angle $\alpha - \frac{\pi}{2}$, or $-\left(\alpha - \frac{\pi}{2}\right)$ with the initial line.

The equations of the asymptotes, therefore, are

$$a\sqrt{e^2-1} = r\cos\left(\theta - \alpha + \frac{\pi}{2}\right),$$

and
$$a\sqrt{e^2-1} = r\cos\left(\theta + \alpha - \frac{\pi}{2}\right)$$

These can be written as

$$\frac{l}{r} = \mp\sqrt{e^2-1}\sin(\theta \mp \alpha)$$

$$= \mp\sqrt{e^2-1}(\sin\theta\cos\alpha \mp \cos\theta\sin\alpha)$$

$$= \frac{\sqrt{e^2-1}}{e}\left(\sqrt{e^2-1}\cos\theta \pm \sin\theta\right)$$

12·4. Equation of the chord of the conic

$$\frac{\mathbf{l}}{\mathbf{r}} = \mathbf{1} + \mathbf{e}\cos\theta \qquad \ldots (1),$$

joining the points whose vectorial angles are $\alpha - \beta$ **and** $\alpha + \beta$.

The equation

$$\frac{l}{r} = A\cos(\theta - \alpha) + B\cos\theta \qquad \ldots (2)$$

represents a straight line in general for it contains two arbitrary constants and will be seen to be of the first degree when converted in Cartesian co-ordinates.

If it passes through the points on the conic whose vectorial angles are $\alpha - \beta$, $\alpha + \beta$, we get on equating the values of r from (1) and (2),

$$1+e\cos(\alpha-\beta)=A\cos\beta+B\cos(\alpha-\beta),$$

and $$1+e\cos(\alpha+\beta)=A\cos\beta+B\cos(\alpha+\beta).$$

From these,

$$A=\sec\beta,\ B=e.$$

The required equation of the chord is, therefore,

$$\mathbf{\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta.}$$

Corollary. If the conic is $\frac{l}{r}=1+e\cos(\theta-\gamma)$, the equation of the chord joining the points $\alpha-\beta$ and $\alpha+\beta$

is $$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

12·41. Tangent. If the points on the conic

$$\frac{l}{r}=1+e\cos\theta$$

whose vectorial angles are $\alpha-\beta$, $\alpha+\beta$ coincide, β becomes zero and the chord in this limiting position becomes a tangent.

The equation of the tangent at the point α of the conic is, therefore,

$$\mathbf{\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta.}$$

Corollary. If the conic is $\frac{l}{r}=1+e\cos(\theta-\gamma)$, the equation of the tangent at the point α is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

Ex. Find the condition that the line $\frac{l}{r}=A\cos\theta+B\sin\theta$ may be a tangent to the conic $\frac{l}{r}=1+e\cos\theta$.

Ans. $(A-e)^2+B^2=1$

12·42. Normal. Writing the equation of the tangent in the form

$$\frac{l}{r}=\cos\theta\,(e+\cos\alpha)+\sin\theta\sin\alpha,$$

the equation of the normal will be of the form

$$\frac{k}{r}=\sin\theta\,(e+\cos\alpha)-\cos\theta\sin\alpha$$
$$=\sin(\theta-\alpha)+e\sin\theta,$$

where k is an arbitrary constant.

If this passes through the point given by

$$\theta=\alpha,\quad \frac{l}{r}=1+e\cos\alpha,$$

we have

$$k\,(1+e\cos\alpha)=l\,e\sin\alpha$$

$$\therefore\qquad k=\frac{e\sin\alpha}{1+e\cos\alpha}\cdot l$$

Hence the equation of the normal at the point α of the conic $\frac{l}{r}=1+e\cos\theta$ is

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{r}=\sin(\theta-\alpha)+e\sin\theta.$$

12·5. Polar. We shall now obtain the equation of the polar of the point (r_1, θ_1) with respect to the conic

$$\frac{l}{r}=1+e\cos\theta.$$

We have seen in Chapter V that the polar of a point is the chord of contact of tangents drawn from that point. We shall use this property to find the polar of (r_1, θ_1).

Let $\alpha-\beta$, $\alpha+\beta$ be the vectorial angles of the points of contact. The equation of the chord then is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta \quad \ldots (1).$$

The equation of the tangent at $\alpha-\beta$ is

$$\frac{l}{r}=\cos(\theta-\alpha+\beta)+e\cos\theta.$$

Since this passes through (r_1, θ_1),

$$\frac{l}{r_1}=\cos(\theta_1-\alpha+\beta)+e\cos\theta_1 \qquad \ldots (2),$$

Similarly $$\frac{l}{r_1}=\cos(\theta_1-\alpha-\beta)+e\cos\theta_1 \qquad \ldots (3).$$

From (2) and (3),

$$\cos(\theta_1-\alpha+\beta)=\cos(\theta_1-\alpha-\beta),$$

i.e., $$\theta_1-\alpha+\beta=\pm(\theta_1-\alpha-\beta).$$

Since $\beta\neq 0$, $$\theta_1-\alpha+\beta=-(\theta_1-\alpha-\beta),$$

i.e., $$\alpha=\theta_1.$$

Therefore from (2) or (3),

$$\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos\beta$$

Substituting in (1), we have

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\theta_1)$$

as the equation of the polar of (r_1, θ_1).

12·6. Solved Examples.

1. *PSP′ is a focal chord of a conic. Prove that*

(*i*) $$\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l},$$

where l is the semi-latus rectum,

(*ii*) *the angle between the tangents at P and P′ is*

$$\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right),$$

where α is the angle between the chord and the major axis.

(*i*) Let the chord PSP' of the conic

$$\frac{l}{r}=1+e\cos\theta \qquad \dots (1)$$

make an angle α with the initial line.

The vectorial angles of P and P' are α and $(\pi+\alpha)$. From (1), therefore

$$\frac{l}{SP}=1+e\cos\alpha,$$

and
$$\frac{l}{SP'}=1+e\cos(\pi+\alpha)$$

$\therefore$
$$\frac{l}{SP}+\frac{l}{SP'}=2,$$

or
$$\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l}.$$

(*ii*) The equations of the tangents at P and P' are

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta,$$

and
$$\frac{l}{r}=-\cos(\theta-\alpha)+e\cos\theta.$$

These can be written as

$$\frac{l}{r}=(\cos\alpha+e)\cos\theta+\sin\alpha\sin\theta,$$

and
$$\frac{l}{r}=(e-\cos\alpha)\cos\theta-\sin\alpha\sin\theta.$$

The slopes being $-\frac{\cos\alpha+e}{\sin\alpha}$ and $-\frac{e-\cos\alpha}{\sin\alpha}$, the angle between the tangents is

$$\tan^{-1}\left(\frac{\frac{e-\cos\alpha}{\sin\alpha}+\frac{\cos\alpha+e}{\sin\alpha}}{1-\frac{(\cos\alpha+e)(e-\cos\alpha)}{\sin^2\alpha}}\right),$$

or
$$\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right)$$

2. *A chord of a conic subtends a constant angle at a focus of the conic. Show that the chord touches another conic.*

Let the conic be

$$\frac{l}{r}=1+e\cos\theta,$$

and let the vectorial angles of the extremities of a chord of this conic be $\alpha-\beta$, $\alpha+\beta$.

The angle which the chord subtends at the pole is 2β, which is constant.

Now the equation of the chord is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta,$$

or
$$\frac{l\cos\beta}{r}=\cos(\theta-\alpha)+e\cos\beta\cos\theta.$$

This evidently touches the conic

$$\frac{l\cos\beta}{r}=1+e\cos\beta\cos\theta,$$

at the point whose vectorial angle is α.

3. *Show that the director circle of the conic*

$$\frac{l}{r}=1+e\cos\theta \text{ is } r^2(1-e^2)+2elr\cos\theta-2l^2=0.$$

The equations of the tangents at the points α and β of the conic

$$\frac{l}{r}=1+e\cos\theta$$

are
$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta,$$

$$\frac{l}{r}=\cos(\theta-\beta)+e\cos\theta.$$

The vectorial angle θ of the point of intersection of these tangents is given by $\cos(\theta-\alpha)=\cos(\theta-\beta)$,

i.e.,
$$\theta-\alpha=\pm(\theta-\beta),$$

Rejecting the plus sign, we get

$$\theta = \frac{\alpha+\beta}{2} \qquad \dots (1)$$

Also from the equation of either tangent,

$$\frac{l}{r} = \cos\frac{\alpha-\beta}{2} + e\cos\theta \qquad \dots (2)$$

where r is the radius vector of the point of intersection.

The equations of the tangents at α, β can be written as

$$\frac{l}{r} = (\cos\alpha + e)\cos\theta + \sin\alpha\sin\theta,$$

and
$$\frac{l}{r} = (\cos\beta + e)\cos\theta + \sin\beta\sin\theta.$$

These are at right angles, if

$$(\cos\alpha + e)(\cos\beta + e) + \sin\alpha\sin\beta = 0,$$

or
$$e^2 + e(\cos\alpha + \cos\beta) + \cos(\alpha-\beta) = 0,$$

i.e.,
$$e^2 + 2e\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\alpha-\beta}{2} - 1 = 0.$$

Substituting from (1) and (2),

$$e^2 - 1 + 2e\cos\theta.\left(\frac{l}{r} - e\cos\theta\right) + 2\left(\frac{l}{r} - e\cos\theta\right)^2 = 0,$$

or
$$r^2(1-e^2) + 2elr\cos\theta - 2l^2 = 0,$$

which is the equation of the director circle.

4. *P, Q, R are three points on the conic*

$$\frac{l}{r} = 1 + e\cos\theta,$$

the focus S being the pole; SP and SR meet the tangent at Q in M and N so that SM = SN = l. Prove that PR touches the conic

$$\frac{l}{r} = 1 + 2e\cos\theta.$$

Let the vectorial angles of P, Q, R be α, γ, β. The equation to PR then is

$$\frac{l}{r}=\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right)+e\cos\theta \qquad \dots (1)$$

The equation to the tangent at Q is

$$\frac{l}{r}=\cos(\theta-\gamma)+e\cos\theta.$$

Two points on this are (l, α) and (l, β).

Therefore,

$$1=\cos(\alpha-\gamma)+e\cos\alpha \qquad \dots (2),$$

and

$$1=\cos(\beta-\gamma)+e\cos\beta \qquad \dots (3).$$

(2) and (3) can be written as

$$(\cos\gamma+e)\cos\alpha+\sin\alpha\sin\gamma=1 \qquad \dots (4),$$

and

$$(\cos\gamma+e)\cos\beta+\sin\beta\sin\gamma=1 \qquad \dots (5).$$

Eliminating $\sin\gamma$ between (4) and (5),

$$(\cos\gamma+e)\sin(\alpha-\beta)=\sin\alpha-\sin\beta,$$

i.e.,

$$\cos\gamma+e=\sec\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \qquad \dots (6).$$

Subtracting (5) from (4)

$$\sin\gamma(\sin\alpha-\sin\beta)+(\cos\gamma+e)(\cos\alpha-\cos\beta)=0,$$

i.e.,

$$\sin\gamma=\sec\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2} \qquad \dots (7).$$

Writing equation (1) as

$$\frac{l}{r}=\sec\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\cos\theta$$
$$+\sec\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\sin\theta+e\cos\theta,$$

and substituting from (6) and (7), the equation to PR is seen to be

$$\frac{l}{r}=\cos(\theta-\gamma)+2e\cos\theta,$$

which touches the conic $\frac{l}{r}=1+2e\cos\theta$, at the point whose vectorial angle is γ.

5. *Show that the equation to the circle circumscribing the triangle formed by the three tangents to the parabola*

$$\frac{2a}{r}=1-\cos\theta$$

drawn at the points whose vectorial angles are α, β and γ, is

$$r=a\operatorname{cosec}\frac{\alpha}{2}\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\gamma}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right),$$

and hence that it always passes through the focus.

The equations of the tangents at α, β, γ are

$$\frac{2a}{r}=\cos(\theta-\alpha)-\cos\theta,$$

$$\frac{2a}{r}=\cos(\theta-\beta)-\cos\theta,$$

$$\frac{2a}{r}=\cos(\theta-\gamma)-\cos\theta.$$

These intersect in pairs in points given by

$$\theta=\frac{\alpha+\beta}{2},\quad \frac{a}{r}=\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

$$\theta=\frac{\beta+\gamma}{2},\quad \frac{a}{r}=\sin\frac{\beta}{2}\sin\frac{\gamma}{2},$$

$$\theta=\frac{\gamma+\alpha}{2},\quad \frac{a}{r}=\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}.$$

By substitution we see that the circle

$$r=a \operatorname{cosec} \frac{\alpha}{2} \operatorname{cosec} \frac{\beta}{2} \operatorname{cosec} \frac{\gamma}{2} \sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right)$$

passes through these points of intersection.

The circle also passes through the focus.

EXAMPLES ON CHAPTER XII

1. Show that the equations

$$\frac{l}{r}=1+e\cos\theta \text{ and } \frac{l}{r}=-1+e\cos\theta$$

represent the same conic.

2. In any conic prove that

(*i*) the tangents at the ends of any focal chord meet on the directrix,

(*ii*) the sum of the reciprocals of two perpendicular focal chords is constant,

(*iii*) the portion of the tangent intercepted between the curve and the directrix subtends a right angle at the corresponding focus.

3. Prove that the locus of the middle point of a focal chord of a conic is another conic of the same kind.

4. PSP' and QSQ' are two perpendicular focal chords of a conic; prove that

$$\frac{1}{PS.SP'}+\frac{1}{QS.SQ'}$$

is constant.

5. If PSQ, PHR be two chords of an ellipse through the foci S, H, then show that

$$\frac{PS}{SQ}+\frac{PH}{HR}$$

is independent of the position of P.

6. The tangents at P and Q to a parabola meet at T. Prove that $ST^2 = SP.SQ$.

7. Chords of a conic subtend a constant angle 2α at the focus. Find the locus of the points where the chords are met by the internal bisectors of the angles that they subtend at the focus.

8. Prove that the locus of the intersection of tangents at the extremities of perpendicular focal radii of a conic is another conic having the same focus.

9. Find the locus of the pole of a chord which subtends a constant angle 2α at a focus of a conic, distinguishing between the cases for which $\cos\alpha \gtreqless e$.

10. Prove that the equation to the locus of the foot of the perpendicular from the focus of the conic $\frac{l}{r} = 1 + e\cos\theta$ on a tangent to it is

$$r^2(e^2-1) - 2ler\cos\theta + l^2 = 0.$$

Discuss the particular case when $e = 1$.

11. A conic is described having the same focus and eccentricity as the conic $\frac{l}{r} = 1 + e\cos\theta$, and the two touch at the point $\theta = \alpha$; prove that the length of its latus rectum is

$$\frac{2l(1-e^2)}{1+2e\cos\alpha+e^2}.$$

12. The normals at α, β, γ on the conic $\frac{l}{r} = 1 + e\cos\theta$ meet in the point whose vectorial angle is φ ; show that

$$2\varphi = \alpha + \beta + \gamma.$$

13. A circle passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are r_1, r_2, r_3, r_4; prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.$$

14. A circle of given radius passing through the focus S of a given conic intersects it in A, B, C, D; show that $SA.SB.SC.SD$ is constant.

15. If the tangent from P subtend the fixed angle β at the focus S, prove that the locus of the midpoint of SP is a conic of eccentricity $e \sec \beta$; and find its latus rectum.
[London, 1944]

16. Two chords QP, PR of a conic subtend equal angles at the focus; prove that the chord QR and the tangent at P intersect on the directrix.

17. Prove that two conics

$$\frac{l_1}{r} = 1 - e_1 \cos \theta,$$

and
$$\frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$$

will touch one another, if

$$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1 l_2(1 - e_1 e_2 \cos \alpha).$$

18. A chord AP of a conic through the vertex A meets the latus rectum in Q, and a parallel chord $P'SQ'$ is drawn through a focus S; prove that the ratio $\dfrac{AP \cdot AQ}{Q'S \cdot SP'}$ is constant.

19. PQ is a chord of an ellipse, one of whose foci is S and PQ passes through a fixed point O. Show that the product $\tan \frac{1}{2} PSO \tan \frac{1}{2} QSO$ is constant.
[London, 1945]

20. An ellipse and a parabola have a common focus S and intersect in two real points P and Q of which P is the vertex of the parabola. If e be the eccentricity of the ellipse and α the angle which SP makes with the major axis, prove that

$$\frac{SQ}{SP} = 1 + \frac{4e^2 \sin^2\alpha}{(1 - e \cos \alpha)^2}.$$

21. If two conics have a common focus, show that two of their common chords will pass through the point of intersection of their directrices.

22. Prove that the two circles which are touched by any circle whose diameter is a focal chord of a given conic have the directrix for their radical axis and the focus for one of their point circles.

Hint. If the conic is $\frac{l}{r}=1+e\cos\theta$, the equations of the two circles will be $r^2(1\pm e)+ler\cos\theta=l^2$.

23. Two parabolas have a common focus and axes opposite, a circle is drawn through the focus touching both parabolas ; prove that

$$3r^{2/3}=a^{2/3}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{2/3},$$

a, b being the latera recta and r the radius of the circle.

CHAPTER XIII

SYSTEMS OF CONICS. CONTACT

13·1. Conic through five points. The general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$, which represents a conic seems to contain six constants a, b, c, f, g, h. On dividing by any one of these constants we see that the number is really five.

If the conic passes through five given points, we shall have five equations to determine the five constants. If the equations are independent, the conic will be determined uniquely. The equations will not be independent if four of the given points lie on a straight line. For, let the line joining the four collinear points be taken as the x—axis and $(0, 0), (\alpha, 0), (\beta, 0)$ $(\gamma, 0)$ the co-ordinates of these points.

We shall now have $c=0$, and

$$a\alpha^2+2g\alpha=0 \quad \ldots \quad (1)$$

$$a\beta^2+2g\beta=0 \quad \ldots \quad (2)$$

$$a\gamma^2+2g\gamma=0 \quad \ldots \quad (3)$$

Since α and β are different from zero, a and g must separately be zero. Equation (3) is then also satisfied, and the conic becomes

$$2hxy+by^2+2fy=0.$$

If the fifth point on the conic be (p, q),

$$2hpq+bq^2+2fq=0,$$

which leaves one of the ratios $h : b : f$ undetermined.

There will, therefore, be an *infinite number of conics passing through five points four of which are collinear.*

If only three points are collinear, the conic will evidently be a pair of straight lines of which one passes through three such points and the other through the remaining two points.

In general we shall have only one conic passing through five points, the exceptional case being when four of the given points are collinear.

13 2. Intersection of two conics. Let the equations to two conics be

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and $$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0.$$

These can be written as

$$ax^2+2x(hy+g)+by^2+2fy+c=0 \qquad \dots \quad (3)$$

and $$a'x^2+2x(h'y+g')+b'y^2+2f'y+c'=0 \qquad \dots \quad (4).$$

If we eliminate x between these equations, we get a fourth degree equation in y. This equation will have four roots, real or imaginary. Also, on eliminating x^2 between (3) and (4) we see that to each value of y there corresponds one value of x. We thus have the following result :

Two conics, in general, intersect in four points, real or imaginary.

Corollary. Two conics cannot touch each other at more than two points. For we can have at most two pairs of concident points of intersection.

13·3. Conics through the points of intersection of two conics.

If $$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and $$S'\equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

be two conics, the equation

$$S+\lambda S'=0 \qquad \dots \quad (1)$$

being of the second degree represents a conic. The values of x and y which simultaneously satisfy $S=0$, $S'=0$ also satisfy $S+\lambda S'=0$, which accordingly passes through the four points of intersection of the two given conics.

The conic (1) has one degree of freedom and represents a conic through four given points. If the fifth point through which (1) passes is also known, the constant λ, hitherto arbitrary, is determined uniquely and equation (1) then represents a fixed conic.

13·4. Double Contact. If $S=0$ be a conic, and

$$lx+my+n=0, \quad l'x+m'y+n'=0$$

two straight lines, the equation

$$S+\lambda(lx+my+n)(l'x+m'y+n')=0$$

represents a conic, which, for any constant value of λ, passes through the four points in which the given lines cut the conic $S=0$.

In particular,

$$S+\lambda(lx+my+n)^2=0 \qquad \dots \quad (1)$$

will be a conic touching the conic $S=0$ at each of the two points in which the line $lx+my+n=0$ meets it. For the conic

$$(lx+my+n)^2=0$$

meets $S=0$ in two pairs of coincident points.

We say that (1) has *double contact* with $S=0$ at the points where the line $lx+my+n=0$ meets it.

If the line $lx+my+n=0$ is a tangent to the conic $S=0$, the conic (1) will touch $S=0$, at four coincident points, and is then said to have *four point contact* or *contact of the third order* with $S=0$.

13.41. Equation of the pair of tangents to a conic.

The pair of tangents from a point (x', y') to the conic $S=0$ can be regarded as the conic which has double contact with $S=0$ where the chord of contact

$$T \equiv axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c = 0$$

meets it.

The equation of the pair of tangents will therefore be

$$S + \lambda T^2 = 0 \qquad \dots \quad (1)$$

where λ is so chosen that (1) passes through (x', y').

This gives

$$S' + \lambda S'^2 = 0,$$

i.e.,

$$\lambda = -\frac{1}{S'}$$

Substituting in (1), we get

$$SS' = T^2.$$

13·5. Some Propositions on Conics.

(1) *All conics passing through the intersections of two rectangular hyperbolas are themselves rectangular hyperbolas.*

If the conics $S = 0$, $S' = 0$ be rectangular hyperbolas, $a + b = 0$, and $a' + b' = 0$.

In the conic $S + \lambda S' = 0$, the sum of the co-efficients of x^2 and y^2 is

$$a + \lambda a' + b + \lambda b',$$

i.e.,

$$a + b + \lambda (a' + b').$$

This is identically equal to zero, since $a + b = 0$ and $a' + b' = 0$.

Hence $S + \lambda S' = 0$ is a rectangular hyperbola, which proves the proposition.

(2) *The common chords of a conic and circle taken in pairs are equally inclined to the axes of the conic.*

If the co-ordinate axes are taken parallel to the axes of the conic, the xy term in the equation of the conic will disappear, and it will have the form

$$ax^2+by^2+2gx+2fy+c=0 \quad (1),$$

a or b being zero if the conic is a parabola.

Now if a circle cuts (1) in four points A, B, C, D and if the equations of any pair of common chords, say AB, CD be

$$lx+my+n=0 \quad \ldots \quad (2),$$

$$l'x+m'y+n'=0 \quad \ldots \quad (3),$$

then

$$ax^2+by^2+2gx+2fy+c$$
$$+\lambda(lx+my+n)(l'x+m'y+n')=0 \quad \ldots \quad (4),$$

represents a conic through the four points in which (2) and (3) meet (1).

By properly choosing λ, (4) will represent a circle.

Now for a circle the co-efficient of xy is zero.

Hence, $$lm'+ml'=0,$$

i.e., $$\frac{l}{m}=-\frac{l'}{m'}$$

The slopes of AB and CD are thus equal in magnitude and opposite in sign showing that AB and CD are equally inclined to the x—axis in opposite directions.

The chords AB, CD are therefore equally inclined to the axes of the conic. The other two pairs of chords AC, BD and AD, BC are similarly equally inclined to the axes.

(3) *Two parabolas can be drawn through four given points of which the axes will be parallel to the centre-locus of the family of conics through the four points.*

Let the line joining two of the points be taken as the axis of x and the joining the other two as the axis of y, and $lx+my=1$, $l'x+m'y=1$ be two lines which cut the axes in the four given points.

The equations of two conics through the four points then are

$$xy=0, \ (lx+my-1)(l'x+m'y-1)=0.$$

The equation

$$\lambda xy+(lx+my-1)(l'x+m'y-1)=0 \quad \ldots \quad (1)$$

therefore represents any conic through the four points.

This can be written as

$$ll'x^2+(lm'+ml'+\lambda)xy+mm'y^2$$
$$-(l+l')x-(m+m')y+1=0 \quad \ldots \quad (2)$$

If (2) is a parabola,

$$4ll'mm'=(lm'+ml'+\lambda)^2.$$

This is a quadratic in λ and has two roots. Two parabolas will therefore pass through the four points. It can be easily seen that the parabolas are real if $ll'mm'$ is positive and imaginary if $ll'mm'$ is negative.

When (2) represents a parabola the second degree terms which then form a perfect square can be written as $(\sqrt{ll'}\,x\pm\sqrt{mm'}\,y)^2$. The axes of the two parabolas are therefore parallel to the lines $\sqrt{ll'}\,x\pm\sqrt{mm'}\,y=0$, or to the line pair

$$ll'x^2-mm'y^2=0 \quad \ldots \quad (3)$$

Now the co-ordinates of the centre of (1) are given by the equations

$$\lambda y+l(l'x+m'y-1)+l'(lx+my-1)=0,$$

$$\lambda x+m(l'x+m'y-1)+m'(lx+my-1)=0.$$

Eliminating λ from these equations and arranging terms,

$$2ll'x^2-2mm'y^2-(l+l')x+(m+m')y=0,$$

which is the locus of the centres of conics through the four points.

The asymptotes of this conic are parallel to the lines

$$ll'x^2-mm'y^2=0,$$

which from (3) are parallel to the axes of the two parabolas through the given points.

(4) *The chord of contact of tangents from a fixed point to a system of conics through four given points passes through a fixed point.*

Let the fixed point be taken as the origin,

and $$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and $$S' \equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$

be two conics which intersect in the four given points.

Then any conic of the system is given by

$$S+\lambda S'=0.$$

The chord of contact of tangents from (0, 0) is

$$gx+fy+c+\lambda(g'x+f'y+c')=0,$$

which passes through the point of intersection of the lines

$$gx+fy+c=0, \quad g'x+f'y+c'=0.$$

(5) *If the chords of contact of two circles with a conic with which they have double contact be parallel, the radical axis of the circles bisects the distance between these chords.*

Let the axes of the conic be taken as the co-ordinate axes, and $ax^2+by^2-1=0$ the equation of the conic.

The equation

$$ax^2+by^2-1+\lambda(lx+my+n)^2=0$$

represents a conic which has double contact with the given conic.

This is a circle, if either $l=0$, or $m=0$,

and $$\lambda(l^2-m^2)=b-a.$$

The chords of contact are thus parallel to one or the other of the axes.

Taking the chords to be parallel to y-axis say, the equations of the circles will be

$$ax^2+by^2-1+(b-a)(x-k_1)^2=0,$$

and $$ax^2+by^2-1+(b-a)(x-k_2)^2=0.$$

The radical axis of these is

$$2x-k_1-k_2=0,$$

which bisects the distance between $x=k_1$ and $x=k_2$

(6) *If the chords of contact of two circles with a conic with which they have double contact be perpendicular, their point of intersection is a limiting point belonging to the coaxal system determined by the two circles.*

Let the conic be $ax^2+by^2-1=0.$

From the preceding example, the equations of the two circles will be

$$S_1 \equiv ax^2+by^2-1+(b-a)(x-\alpha)^2=0,$$

and $$S_2 \equiv ax^2+by^2-1-(b-a)(y-\beta)^2=0.$$

Now $S_1-S_2 \equiv (b-a)\{(x-\alpha)^2+(y-\beta)^2\}=0$

is a circle belonging to the coaxal system determined by $S_1=0$, $S_2=0$; and, being a point circle, is the limiting point (α, β) which is also the point of intersection of the chords of double contact.

13.6. Equation of the conic which touches four fixed straight lines.

Let two of the lines be taken as the co-ordinate axes and $lx+my=1$, $l'x+m'y=1$ be the equations of the other two lines.

The conic touching the co-ordinate axes is

$$(ax+by-1)^2+2\lambda xy=0 \qquad \ldots \quad (1).$$

The lines joining the origin to the points in which the line $lx+my=1$ cuts (1) are given by

$$(ax+by-lx-my)^2+2\lambda xy=0$$

These lines are coincident, if

$$(a-l)^2(b-m)^2=\{(a-l)(b-m)+\lambda\}^2$$

i.e., if $$\lambda=-2(a-l)(b-m).$$

The line $lx+my=1$ now touches the conic.

Equation (1) therefore represents a conic touching the lines

$x=0$, $y=0$, $lx+my=1$, and $l'x+m'y=1$, if

$$\lambda=-2(a-l)(b-m)=-2(a-l')(b-m').$$

Ex. 1. Prove that the conic passing through the points $(a, 0)$, $(b, 0)$, $(0, c)$, $(0, d)$, (b, d), where a, b, c, d are positive and $b>a$, $d>c$, is an ellipse. If $b=2a$, $d=2c$, prove that the centre is the point $\left(\frac{6a}{5}, \frac{6c}{5}\right)$.

Ex. 2. A circle cuts the parabola $y^2=4ax$ in the points 't_1', 't_2', 't_3', 't_4'. Prove that $t_1+t_2+t_3+t_4=0$.

✗ **Ex. 3.** Prove that the conics

$$2x^2+y^2-1=0,$$

and $$18x^2+24xy+10y^2-8x-6y=0$$

touch each other at two distinct points. Find the co-ordinates of the intersection of the tangents at these points.

Ans. (2, 3).

Ex. 4. Find the equation of the parabola which touches the conic

$$x^2+xy+y^2-2x-2y+1=0$$

at the points where it is cut by the line $x+y+1=0$.

Determine the equation of the axis and the co-ordinates of the focus of this parabola.

Ans. $(x-y)^2=14x+14y-1$; $x-y=0$; $(\frac{25}{14}, \frac{25}{14})$.

Ex. 5. Prove that the locus of the centres of conics which touch the co-ordinate axes at distances a and b from the origin is the straight line $ay=bx$.

The conics have double contact with $xy=0$ at the extremities of the chord $\frac{x}{a}+\frac{y}{b}=1$.

The equation

$$\lambda xy+\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=0$$

represents the system.

The centre is given by

$$\lambda y+\frac{2}{a}\left(\frac{x}{a}+\frac{y}{b}-1\right)=0,$$

$$\lambda x+\frac{2}{b}\left(\frac{x}{a}+\frac{y}{b}-1\right)=0.$$

Eliminating λ,

$$\frac{y}{x}=\frac{b}{a}$$

i.e., $$ay=bx$$

is the required locus.

Ex. 6. PQ is a chord of an ellipse and T its pole. Show that T is a limiting point of the coaxal family of circles fixed by the circle on PQ as diameter and the director circle.

[I. C. S., 1939]

Ex. 7. Show that the locus of the centres of rectangular hyperbolas which have four point contact with a given parabola is an equal parabola having the same axis and directrix.

Ex. 8. Prove that the locus of the poles of PQ with respect to all conics passing through four fixed points P, Q, R, S is a straight line.

Ex. 9. $S=0$ is the equation of a conic, $L=0$ is the equation of a line meeting the conic in two points P and Q, and $T=0$ is the equation of the tangent to the conic at P. Interpet the equations

(*i*) $S-\lambda L^2=0$, (*ii*) $S-\lambda LT=0$, (*iii*) $S-\lambda T^2=0$, where λ is a parameter.

A variable conic passes through two fixed points A, B and has double contact with a fixed conic. Prove that the chord of contact passes through one or other of two fixed points on AB.

[Math. Tripos, 1939]

Ex. 10. Two conics have double contact. Prove that the polar of the centre of one conic with respect to the other is parallel to the common chord.

Ex. 11. The conics $S_1=0$, $S_2=0$ have a pair of common chords $u=0$, $v=0$ such that $S_1-S_2\equiv uv$. Prove that the conic

$$\lambda^2u^2-2\lambda(S_1+S_2)+v^2=0$$

has double contact with both $S_1=0$ and $S_2=0$.

A conic has finite double contact with each of the conics

$$x^2+y^2-e^2(x+c)^2=0,\ x^2+y^2-e'^2(x+c)^2=0.$$

Prove that the chords of contact are perpendicular chords through the origin; also that if $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$ all such conics are rectangular hyperbolas.

Ex. 12. Show that the centres of all conics which touch four fixed straight lines lie on a straight line.

13·7. Foci of Central Conics. The equation of a central conic referred to its principal axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots (1),$$

which represents a hyperbola if b^2 is negative.

The co-ordinates of the foci of this conic are $(\pm\sqrt{a^2-b^2}, 0)$, the corresponding directrices being

$$x=\pm\frac{a^2}{\sqrt{a^2-b^2}}.$$

From the symmetry of equation (1) it is apparent that there are two other foci with co-ordinates $(0, \pm\sqrt{b^2-a^2})$. These foci are imaginary since $b^2=a^2(1-e^2)$, e being the eccentricity.

The corresponding directrices are

$$y=\pm\frac{b^2}{\sqrt{b^2-a^2}}.$$

It can be seen that each directrix is the polar of the corresponding focus.

A parabola being the limiting case of an ellipse may also be regarded as having four foci, three of which lie at infinity.

We thus have the following result :

Every central conic has four foci, two real and two imaginary. The real foci lie on one axis of the conic and the imaginary ones on the other axis.

Ex. Show that every central conic has two eccentricities, of which one is real and the other imaginary if the conic is an ellipse, and both are real if the conic is a hyperbola.

13·71. Tangents from a focus. The equation to the pair of tangents from the focus $(\sqrt{a^2-b^2}, 0)$ to the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is}$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2-b^2}{a^2}-1\right)=\left(\frac{x\sqrt{a^2-b^2}}{a^2}-1\right)^2,$$

which on simplification gives

$$x^2-2x\sqrt{a^2-b^2}+(a^2-b^2)+y^2=0,$$

that is

$$(x-\sqrt{a^2-b^2})^2+y^2=0$$

which is a point circle at the focus $(\sqrt{a^2-b^2}, 0)$ itself.

It can similary be seen that the pairs of tangents from each of the three remaining foci are point circles at the corresponding foci.

13·72. Foci of a conic referred to rectangular axes.

Let $$ax^2+2hxy+by^2+2gx+2fy+c=0$$

be the equation to a conic when the axes are rectangular, and let (x', y') be the co-ordinates of a focus.

The equation to the pair of tangents from (x', y') is

$$(ax^2+2hxy+by^2+2gx+2fy+c)(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)=\{x(ax'+hy'+g)+y(hx'+by'+f)+(gx'+fy'+c)\}^2.$$

Since this must reduce to a point circle at (x', y'), the co-efficients of x^2 and y^2 must be equal and the co-efficient of xy must vanish.

This gives

$$(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)(a-b)$$
$$-(ax'+hy'+g)^2+(hx'+by'+f)^2=0,$$

and
$$h(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)$$
$$-(ax'+hy'+g)(hx'+by'+f)=0.$$

From these we get

$$\frac{(\mathbf{ax'+hy'+g})^2-(\mathbf{hx'+by'+f})^2}{\mathbf{a-b}}$$
$$=\frac{(\mathbf{ax'+hy'+g})(\mathbf{hx'+by'+f})}{\mathbf{h}}$$
$$=\mathbf{ax'^2+2hx'y'+by'^2+2gx'+2fy'+c}$$

as the equations to determine the co-ordinates of the foci of the general conic.

13·73. Axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \dots (1)$$

We have seen in § 13·7 that the foci of a conic lie upon its axes and we also know that the axes pass through the centre of the conic.

From the result of the preceding article, the conic whose equation is

$$\frac{(ax+hy+q)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h} \quad (2)$$

passes through the foci of the conic represented by the general equation (1). Besides, equation (2) is satisfied by the co-ordinates of the centre of (1), for the centre is given by the equations

$$ax+hy+g=0,$$

and

$$hx+by+f=0.$$

Hence (2) passes through four foci and the centre of (1); and since the axes of (1) are the only conic which pass through these five points, not more than three of which are collinear, equation (2) represents the axes of (1).

13·74 Directrices of the conic.

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots (1)$$

The directrices of a conic being the polars of, or which is the same as the chords of contact of tangents from, the corresponding foci, equation (1) is equivalent to

$$(x-x')^2+(y-y')^2-(lx+my+n)^2=0,$$

where $lx+my+n=0$ is the equation of the directrix corresponding to the focus (x', y').

Comparing co-efficients,

$$\frac{l^2-1}{a}=\frac{lm}{h}=\frac{m^2-1}{b}=\frac{ln+x'}{g}=\frac{mn+y'}{f}$$
$$=\frac{n^2-x'^2-y'^2}{c}=\lambda \text{ say.}$$

From these we get

$$\lambda(ax'+hy'+g)=l(lx'+my'+n) \quad \dots (2),$$

$$\lambda(hx'+by'+f)=m(lx'+my'+n) \quad \dots (3),$$

$$\lambda(gx'+fy'+c)=n(lx'+my'+n) \quad \dots (4),$$

and

$$\lambda(a-b)=l^2-m^2 \quad \dots (5),$$

$$\lambda h=lm \quad \dots (6).$$

Eliminating x', y' from (2), (3) and (4),

$$\begin{vmatrix} \lambda a-l^2 & \lambda h-lm & \lambda g-ln \\ \lambda h-lm & \lambda b-m^2 & \lambda f-mn \\ \lambda g-ln & \lambda f-mn & \lambda c-n^2 \end{vmatrix}=0,$$

as the cubic to determine λ. Two roots of this will be seen to be zero. The third non-zero root when substituted in (5) and (6) will give the ratios $l:m:n$ which will determine the *directrices*.

Ex. 1. Prove that the general equation of conics having the points (a, b) and $(-a, -b)$ as foci may be written

$$(x^2-a^2-\lambda)(y^2-b^2-\lambda)=(xy-ab)^2.$$

Ex. 2 Determine the foci and directrices of the conic whose equation is $x^2+12xy-4y^2-6x+4y+9=0$

Ans. Real foci $(1, -1), (-1, 2)$.

Real directrices $2x-3y+4=0$ *and* $2x-3y-1=0$.

13·8. Circle of Curvature.

Let $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \dots (1)$

be the equation of a conic, and

$$T\equiv axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0$$

the equation of the tangent at the point (x', y') of (1).

The equation

$$S+T(lx+my+n)=0 \qquad \dots (2)$$

represents a conic which meets (1) in points where $T=0$

and $$lx+my+n=0 \qquad \dots (3)$$

meet it.

But $T=0$ meets (1) in two coincident points.

The conics (1) and (2) accordingly touch each other at $(x' \; y')$ which constitutes a pair of coincident points for both of them.

Now let (3) pass through (x', y') in which case it has the form

$$l(x-x')+m(y-y')=0.$$

Equation (2) thus becomes

$$S+T\{l(x-x')+m(y-y')\}=0 \qquad \dots (4)$$

which represents a conic having three point contact at (x', y') with (1).

If the co-efficients in (4) are so chosen that it represents a circle, then this circle is called the **circle of curvature** or the **osculating circle** of the conic $S=0$ at the point (x', y').

Thus *the circle of curvature of a conic at a given point has three point contact with the conic at that point.*

This is the limiting case in which two adjacent chords of intersection of a circle and a conic coincide with the tangent at the point, that is, the circle and the conic have two consecutive tangents in common.

Defining curvature as the rate of deflection of the tangent at any point of a curve we see that *the curvature of a conic at any point is equal to the reciprocal of the radius of the circle of curvature, or as is technically termed, the radius of curvature of the conic at that point.*

The *centre of curvature* at any point is the centre of the circle of curvature at that point.

The solved examples in the following set will serve as useful illustrations of these ideas.

Ex. 1 Find the equation of the circle of curvature and the co-ordinates of the centre of curvature at the point $(at^2, 2at)$ of the parabola $y^2=4ax$.

Show also that the circle of curvature meets the parabola again in the point $(9at^2, -6at)$.

The equation of the tangent at $(at^2, 2at)$ is

$$ty=x+at^2,$$

or

$$y-\frac{x}{t}-at=0.$$

Since the chords of intersection of a conic and a circle taken in pairs, are equally inclined to the axes of the conic, the equation to the common chord of the parabola and its circle of curvature at $(at^2, 2at)$ will be

$$y+\frac{x}{t}=2at+\frac{at^2}{t}=3at$$

The circle of curvature will thus be

$$y^2-4ax+\lambda\left(y-\frac{x}{t}-at\right)\left(y+\frac{x}{t}-3at\right)=0,$$

provided λ is so chosen that the co-efficients of x^2 and y^2 become equal to each other.

This gives $\qquad 1+\lambda=-\frac{\lambda}{t^2}$, i.e., $\lambda=-\frac{t^2}{1+t^2}$.

The equation of the circle of curvature is therefore

$$y^2-4ax-\frac{t^2}{1+t^2}\left(y-\frac{x}{t}-at\right)\left(y+\frac{x}{t}-3at\right)=0,$$

that is

$$(1+t^2)(y^2-4ax)-t^2\left(y^2-\frac{x^2}{t^2}-4aty+2ax+3a^2t^2\right)=0,$$

or $$x^2+y^2-2ax(2+3t^2)+4at^3y-3a^2t^4=0.$$

Writing this as

$$\{x-a(2+3t^2)\}^2+(y+2at^3)^2=a^2\{(2+3t^2)^2+4t^6+3t^4\}$$
$$=4a^2(1+t^2)^3,$$

we see that the co-ordinates of the centre of curvature are

$$\{a(2+3t^2),\ -2at^3\}.$$

and the radius of curvature is $2a(1+t^2)^{3/2}$.

For the second part, let $(at'^2, 2at')$ be the co-ordinates of the point in which the circle of curvature again meets the parabola.

Then

$$y+\frac{x}{t}=3at$$

passes through $(at'^2, 2at')$,

Hence $$2at'+\frac{at'^2}{t}=3at,$$

or $$t'^2+2t't-3t^2=0,$$

from which $t'=t$, or $-3t$.

The other point of intersection is therefore $(9at^2, -6at)$.

Ex. 2. Show that the radius of curvature at a point P $(a\cos\varphi, b\sin\varphi)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{CD^3}{ab}$ where CD is the semi-diameter conjugate to CP.

If θ be the eccentric angle of the point in which the circle of curvature again cuts the ellipse, show that

$$\varphi=\tfrac{1}{3}(2n\pi-\theta).$$

The equation of the tangent at P is

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1.$$

The common chord of the ellipse and its circle of curvature is

$$\frac{x}{a}\cos\varphi-\frac{y}{b}\sin\varphi=\cos^2\varphi-\sin^2\varphi=\cos 2\varphi.$$

The circle of curvature will now be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi-1\right)$$
$$\left(\frac{x}{a}\cos\varphi-\frac{y}{b}\sin\varphi-\cos 2\varphi\right)=0 \quad \ldots \quad (1),$$

λ being given by

$$\frac{1}{a^2}+\frac{\lambda}{a^2}\cos^2\varphi=\frac{1}{b^2}-\frac{\lambda\sin^2\varphi}{b^2}.$$

From this,
$$\lambda=\frac{a^2-b^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}.$$

Substituting this value of λ in (1) and multiplying out by $a^2\sin^2\varphi+b^2\cos^2\varphi$,

$$(a^2\sin^2\varphi+b^2\cos^2\varphi)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$
$$+(a^2-b^2)\left\{\frac{x^2}{a^2}\cos^2\varphi-\frac{y^2}{b^2}\sin^2\varphi-\frac{x}{a}\cos\varphi(1+\cos 2\varphi)\right.$$
$$\left.+\frac{y}{b}\sin\varphi(1-\cos 2\varphi)+\cos 2\varphi\right\}=0.$$

This simplifies to

$$x^2+y^2-\frac{2(a^2-b^2)}{a}\cos^3\varphi . x+\frac{2(a^2-b^2)}{b}\sin^3\varphi . y$$
$$=a^2\sin^2\varphi+b^2\cos^2\varphi-(a^2-b^2)\cos 2\varphi,$$

which can be written as

$$\left(x-\frac{a^2-b^2}{a}\cos^3\varphi\right)^2+\left(y+\frac{a^2-b^2}{b}\sin^3\varphi\right)^2$$
$$=\frac{(a^2\sin^2\varphi+b^2\cos^2\varphi)^3}{a^2b^2}.$$

The radius of curvature is therefore

$$\frac{(a^2\sin^2\varphi+b^2\cos^2\varphi)^{3/2}}{ab},$$

which is the same as $\dfrac{CD^{3/2}}{ab}$ where D is an extremity of the diameter conjugate to CP.

Further, if $(a\cos\theta, b\sin\theta)$ be the point in which the ellipse is again cut by the circle of curvature, this point lies on the common chord

$$\frac{x}{a}\cos\varphi - \frac{y}{b}\sin\varphi = \cos 2\varphi.$$

Hence, $\cos\theta\cos\varphi - \sin\theta\sin\varphi = \cos 2\varphi,$

i.e., $$\cos(\theta+\varphi) = \cos 2\varphi,$$

i.e., $$\sin\frac{\theta+3\varphi}{2}\sin\frac{\theta-\varphi}{2} = 0.$$

Since $$\theta - \varphi \neq 2n\pi, \frac{\theta+3\varphi}{2} = n\pi,$$

or $$\varphi = \tfrac{1}{3}(2n\pi - \theta).$$

Ex. 3. The circles of curvature of a fixed parabola at the extremities of a focal chord meet the parabola again at H and K. Prove that HK passes through a fixed point.

[I. C. S., 1937]

Ex. 4. The circles of curvature at three points P, Q, R of an ellipse all cut the ellipse again in the same point, prove that the centre of the ellipse is the centre of mean position of P, Q, R.

Hint. The circles of curvature at the points $\frac{1}{3}(2\pi-\theta)$, $\frac{1}{3}(4\pi-\theta)$, $\frac{1}{3}(6\pi-\theta)$ cut the ellipse again in the same point θ.

Ex. 5. ρ and ρ' are the radii of curvature at the ends P and D of conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ;$$

prove that

$$\rho^{2/3}+\rho'^{2/3}=\frac{a^2+b^2}{a^{2/3}b^{2/3}},$$

and that the locus of the middle point of the line joining the centres of curvature at P and D is

$$(ax+by)^{2/3}+(ax-by)^{2/3}=(a^2-b^2)^{2/3}.$$

13·9. The Line at infinity. We have seen in Chapter II (§ 2·33) that $c=0$ represents a straight line lying wholly at infinity. We shall now interpret a few equations associated with the line at infinity.

1. $S=\lambda u$, λ *being a constant,* $S=0$ *and* $u=0$ *the equations of a conic and line respectively.*

Since $\lambda=0$ is the equation of the line at infinity, $S=\lambda u$ is a conic passing through the points where the conic $S=0$ is cut by $u=0$ and the line at infinity. The two conics thus have the same intersections with the line at infinity. Their asymptotes are parallel.

As a particular case let us consider the circle

$$x^2+y^2-2gx-2fy-c=0 \qquad \dots \quad (1)$$

Writing the equation as

$$x^2+y^2-a^2=2gx+2fy+c-a^2,$$

we see that it has the form $S=\lambda u$.

The circles (1) and $x^2+y^2-a^2=0$ therefore meet the line at infinity in the same two (imaginary) points. These are called the *Circular Points at Infinity.*

2. $S=\lambda$.

Writing the equation as $S=\lambda(0.x+0.y+1)^2$, we see that the conic $S=0$ and $S=\lambda$ have double contact at the points where they meet the line at infinity. From the definition of an asymptote it is therefore apparent that the two conics have the same pair of real or imaginary asymptotes.

In particular, if $S=0$ is a circle, $S=\lambda$ is a concentric circle, and the two have double contact at the circular points at infinity.

EXAMPLES ON CHAPTER XIII

1. A circle and a rectangular hyperbola intersect in four points, and one of their common chords is a diameter of the hyperbola; show that the other chord is a diameter of the circle.

2. Prove that in general two parabolas can be drawn through the points of intersection of the conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and $$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$

and that their axes will be at right angles if

$$\frac{h}{a-b}=\frac{h'}{a'-b'}.$$

3. Prove that the locus of a point, the sum or difference of the tangents from which to two given circles is constant is a conic having double contact with each of the two circles.

4. A circle is drawn to touch the parabola $y^2=4ax$ at the point P and to cut it in the origin and in the further point Q. Prove that PQ touches the parabola

$$y^2+32ax=0.$$ [London, 1945]

5. Parabolas are drawn with a fixed focus S, having a fixed line through S as their axis. Show that two such parabolas pass through any point not lying on the common axis, and that they intersect at right angles.

[Math. Tripos, 1938]

6. A rectangular hyperbola has double contact with a parabola; prove that the centre of the hyperbola and the pole of the chord of contact will be equidistant from the directrix of the parabola.

7. A rectangular hyperbola has double contact with a fixed central conic. If the chord of contact always passes through a fixed point, prove that the locus of the centre of the rectangular hyperbola is a circle passing through the centre of the fixed conic.

8. A conic has double contact with the parabola $y^2=4ax$. If the chord of contact passes through the vertex and the conic passes through the focus of the parabola, prove that the locus of the centre of the conic is the parabola $y^2=a(2x-a)$.

9. Prove that all chords of a conic which subtend a right angle at a fixed point P on the conic, pass through a fixed point F on the normal at P. What can be deduced when the conic is a rectangular hyperbola?

When the conic is a parabola, prove that PF is bisected by the axis. [Wales, 1945]

10. P and Q are two points in the plane of a conic S. Prove that the locus of intersection of pairs of lines through P and Q which are conjugate with respect to S is a conic C through P and Q and that the line PQ has the same pole in S and C.

11. A family of conics have double contact with a given conic at the extremities of a given chord. Show that the locus of the centres of conics of the family is the diameter of the given conic conjugate to the given chord.

12. A parabola, of latus rectum l, touches a fixed equal parabola, the axes of the two curves being parallel; prove that the vertex of the moving curve lies on a parabola of latus rectum $2l$.

13. Two circles each have double contact with a parabola and touch each other. Prove that the difference between their radii is equal to the latus rectum of the parabola. [Math. Tripos, 1947]

14. A circle touches a hyperbola at two points, the chord of contact being parallel to the transverse axis. Prove that the ratio of the length of the tangent to the

circle from any point of the conic to the distance of the point from the chord of contact is equal to the eccentricity of the conjugate hyperbola.

15. Any tangent drawn to the circle of curvature at the vertex of the parabola $y^2=4ax$ meets the parabola in points whose ordinates are y_1 and y_2, show that

$$\frac{1}{y_1}-\frac{1}{y_2}=\frac{1}{2a}.$$

16. Prove that the locus of the pole of the axis of x with respect to the circle of curvature at any point of the parabola

$$y^2=4ax$$

is

$$(x-2a)^3y^2=12a(x^2-ax+a^2)^2.$$

17. Three points $(at_r^2, 2at_r)$, $r=1, 2, 3$, on the parabola $y^2=4ax$ are such that their centres of curvature are collinear, prove that $\Sigma\frac{1}{t_r}=0$.

18. The circle of curvature of the rectangular hyperbola $x^2-y^2=a^2$ at the point $(a \operatorname{cosec} \theta, a \cot \theta)$ meets the curve again in the point $(a \operatorname{cosec} \varphi, a \cot \varphi)$. Show that

$$\tan\frac{\varphi}{2}\tan^3\frac{\theta}{2}=1.$$

19. Circles of curvature are drawn to a hypebola and its conjugate at the ends of conjugate diameters ; prove that their radical axis is parallel to one of the asymptotes.

20. The circle of curvature of a parabola at P meets the parabola again in Q and QL, QM are drawn tangents to the circle and parabola at Q, each terminated by the other curve ; prove that when LM subtends a right angle at P, PL is parallel to the axis, and that this is the case when the focal distance of P is one-third of the latus rectum.

21. Prove that the locus of the centres of non-degenerate conics having four point contact with the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at $(a\cos\theta, b\sin\theta)$ is the line $bx\sin\theta - ay\cos\theta = 0$.

22. A circle of given radius cuts an ellipse in four points; prove that the continued product of the diameters of the ellipse parallel to the common chords is constant.

23. Prove that the axis of a parabola which passes through the feet of the four normals drawn to a given ellipse from a given point will be parallel to one of the equi-conjugate diameters of the ellipse.

Hint. If (x', y') be the given point and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

the given ellipse, the equation of the axis of the parabola will be

$$\frac{1}{a}\left(x+\frac{a^4x'}{a^4-b^4}\right)\pm\frac{1}{b}\left(y+\frac{b^4y'}{a^4-b^4}\right)=0.$$

24. A triangle ABC is formed by the line-pair $ax^2+2hxy+by^2=0$ and the line $x+y+1=0$ the vertex A being at the origin. Prove that

$$2(ax^2+2hxy+by^2)-(a+b)(x+y+1)^2=0$$

is a rectangular hyperbola touching AB at B and AC at C; and that

$$(a+b-2h)(x^2+y^2)+(b-a-2h)x+(a-b-2h)y=0$$

is the circum-circle of the triangle ABC. [Oxford, 1946]

25. Prove that through any four points one rectangular hyperbola and two parabolas can be drawn.

Prove that the asymptotes of the rectangular hyperbola bisect the angles between the directions of the axes of the parabolas.

If these axes be at right angles, prove that the four points lie on a circle.

CHAPTER XIV

CONFOCAL CONICS

14·1. Equation of confocals. Conics having the same foci are said to form a confocal system.

In §13.7 we have seen that the foci of a central conic lie on its principal axes. All confocal central conics will therefore have a common centre and the same lines as principal axes. The general equation of conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will, accordingly, have the form

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1 \qquad \dots \quad (1).$$

The distance of a focus from the centre being the same for all confocals, we have

$$a_1^2-b_1^2=a^2-b^2=\text{constant},$$

i.e., $\qquad a_1^2-a^2=b_1^2-b^2=\lambda$, say.

The general equation (1) now becomes

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

which for different values of λ represents different conics confocal with the ellipse.

14·2. Confocals through a given point. Let

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad \dots \quad (1)$$

be a conic confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If (1) passes through (x', y'),

$$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2+\lambda}=1,$$

that is

$$f(\lambda)\equiv(a^2+\lambda)(b^2+\lambda)-x'^2(b^2+\lambda)-y'^2(a^2+\lambda)=0 \quad \cdots \quad (2).$$

For $\lambda=+\infty$, $f(\lambda)$ is $+ve$,

for $\lambda=-b^2$, $f(\lambda)$ is $-ve$,

for $\lambda=-a^2$, $f(\lambda)$ is $+ve$,

a being greater than b.

One of the roots of (2) therefore lies between $+\infty$ and $-b^2$, and the other between $-b^2$ and $-a^2$.

The root lying between $+\infty$ and $-b^2$ makes both $a^2+\lambda$ and $b^2+\lambda$ positive and the one lying between $-b^2$ and $-a^2$ makes $a^2+\lambda$ positive and $b^2+\lambda$ negative.

We thus see that *through any point in the plane of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *there pass two confocals, one of which is an ellipse and the other a hyperbola.*

14·3. The co-ordinates of any point in the plane of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of the parameters λ_1, λ_2 of the confocals through it.

Let (x', y') be any point in the plane of the ellipse.

Then

$$\frac{x'^2}{a^2+\lambda_1}+\frac{y'^2}{b^2+\lambda_1}=1,$$

and

$$\frac{x'^2}{a^2+\lambda_2}+\frac{y'^2}{b^2+\lambda_2}=1.$$

These give

$$\frac{x'^2}{\frac{1}{b^2+\lambda_2}-\frac{1}{b^2+\lambda_1}}=\frac{y'^2}{\frac{1}{a^2+\lambda_2}-\frac{1}{a^2+\lambda_1}}$$

$$=\frac{1}{\frac{1}{(a^2+\lambda_1)(b^2+\lambda_2)}-\frac{1}{(a^2+\lambda_2)(b^2+\lambda_1)}}$$

$$=\frac{(a^2+\lambda_1)(a^2+\lambda_2)(b^2+\lambda_1)(b^2+\lambda_2)}{(\lambda_1-\lambda_2)(a^2-b^2)},$$

i.e., $$x'^2=\frac{(a^2+\lambda_1)(a^2+\lambda_2)}{a^2-b^2},$$

and $$y'^2=\frac{(b^2+\lambda_1)(b^2+\lambda_2)}{b^2-a^2}.$$

Example. Find the conics confocal with $x^2+2y^2=2$ which pass through the point (1, 1).

Ans. $3x^2-y^2\pm\sqrt{5}(x^2-y^2)=2$.

14·4. Solved Examples.

1. *Show that the confocal hyperbola through the point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *whose eccentric angle is* α *has for equation*

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2.$$

If λ be the parameter of the confocal through $(a\cos\alpha, b\sin\alpha)$,

$$\frac{a^2\cos^2\alpha}{a^2+\lambda}+\frac{b^2\sin^2\alpha}{b^2+\lambda}=1,$$

i.e., $(b^2+\lambda)a^2\cos^2\alpha+(a^2+\lambda)\,b^2\sin^2\alpha=(a^2+\lambda)(b^2+\lambda),$

i.e., $$\lambda^2+\lambda(b^2\cos^2\alpha+a^2\sin^2\alpha)=0,$$

i.e., $$\lambda=0,\text{ or } -(a^2\sin^2\alpha+b^2\cos^2\alpha).$$

The equation to the confocal therefore is

$$\frac{x^2}{a^2-(a^2\sin^2\alpha+b^2\cos^2\alpha)}+\frac{y^2}{b^2-(a^2\sin^2\alpha+b^2\cos^2\alpha)}=1,$$

or $$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2.$$

2. *If the confocals through* (x_1, y_1) *to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

show that (i) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=\frac{\lambda_1\lambda_2}{a^2b^2},$

and (ii) $x_1^2+y_1^2-a^2-b^2=\lambda_1+\lambda_2.$

The conic confocal with the given ellipse is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Since this passes through (x_1, y_1),

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1,$$

i.e., $\lambda^2+\lambda(a^2+b^2-x_1^2-y_1^2)+a^2b^2\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)=0.$

Since λ_1, λ_2 are the roots of this quadratic,

$$\lambda_1\lambda_2=a^2b^2\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)$$

$$\lambda_1+\lambda_2=x_1^2+y_1^2-a^2-b^2,$$

which give the desired results.

3. *Prove that the two conics*
$ax^2+2hxy+by^2=1$ *and* $a'x^2+2h'xy+b'y^2=1$

can be placed so as to be confocal, if

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Let the conics $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$ referred to their principal axes be respectively

$$\alpha x^2+\beta y^2=1 \text{ and } \alpha' x^2+\beta' y^2=1.$$

If the conics can be placed so as to be confocal,

$$\frac{1}{\alpha}-\frac{1}{\beta}=\frac{1}{\alpha'}-\frac{1}{\beta'},$$

or

$$\frac{\beta-\alpha}{\alpha\beta}=\frac{\beta'-\alpha'}{\alpha'\beta'} \qquad \ldots \quad (1)$$

Now by invariants,

$$\left.\begin{aligned}\alpha+\beta=a+b, \quad \alpha\beta=ab-h^2;\\ \alpha'+\beta'=a'+b', \; \alpha'\beta'=a'b'-h'^2.\end{aligned}\right\} \qquad \ldots \quad (2)$$

Writing (1) as

$$\frac{(\alpha+\beta)^2-4\alpha\beta}{\alpha^2\beta^2}=\frac{(\alpha'+\beta')^2-4\alpha'\beta'}{\alpha'^2\beta'^2}$$

and substituting from (2),

$$\frac{(a+b)^2-4(ab-h^2)}{(ab-h^2)^2}=\frac{(a'+b')^2-4(a'b'-h'^2)}{(a'b'-h'^2)^2},$$

which is the same as

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

14·5. Intersection of confocals. We shall show below that the two confocals through any point intersect at right angles.

Let the confocals through (x', y') be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

Since (x', y') lies on both,

$$\frac{x'^2}{a^2+\lambda_1}+\frac{y'^2}{b^2+\lambda_1}=1, \text{ and } \frac{x'^2}{a^2+\lambda_2}+\frac{y'^2}{b^2+\lambda_2}=1.$$

Subtracting one from the other and removing the common factor $\lambda_1-\lambda_2$,

$$\frac{x'^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y'^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0,$$

which is the condition that the tangents

$$\frac{xx'}{a^2+\lambda_1}+\frac{yy'}{b^2+\lambda_1}=1, \text{ and } \frac{xx'}{a^2+\lambda_2}+\frac{yy'}{b^2+\lambda_2}=1$$

to the two confocals at the common point (x', y') be at right angles.

We thus see that *two confocals intersect orthogonally at their common points.*

14·6. Some Propositions on Confocals.

(1) *The difference of the squares of the perpendiculars drawn from the centre on any two parallel tangents to two given confocals is constant.*

Let the lines

$$x \cos \alpha + y \sin \alpha - p = 0, \qquad x \cos \alpha + y \sin \alpha - p' = 0$$

touch the confocals

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

respectively.

Then (§ 9·4, Ex.)

$$p^2=(a^2+\lambda_1)\cos^2\alpha+(b^2+\lambda_1)\sin^2\alpha,$$

and $$p'^2=(a^2+\lambda_2)\cos^2\alpha+(b^2+\lambda_2)\sin^2\alpha.$$

Subtracting.

$$p^2-p'^2=\lambda_1-\lambda_2,$$

which proves the proposition.

(2) *One conic and only one of a confocal system will touch a given straight line.*

Let the line $x \cos \alpha + y \sin \alpha = p$ be a tangent to the confocal

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1.$$

Then $\quad (a^2+\lambda) \cos^2 \alpha + (b^2+\lambda) \sin^2 \alpha = p^2,$

which gives only one value of λ. Hence one and only one conic of the confocal system will touch the given line.

(3) *The point of intersection of two perpendicular tangents one to each of two given confocals lies on a circle.*

Let the confocals be

$$\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1, \text{ and } \frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1.$$

The perpendicular straight lines

$$x \cos \alpha + y \sin \alpha = \sqrt{\{(a^2+\lambda_1) \cos^2 \alpha + (b^2+\lambda_1) \sin^2 \alpha\}}$$

and $x \sin \alpha - y \cos \alpha = \sqrt{\{(a^2+\lambda_2) \sin^2 \alpha + (b^2+\lambda_2) \cos^2 \alpha\}}$

touch the conics respectively.

Squaring and adding the two sides of these equations,

$$x^2+y^2 = \lambda_1+\lambda_2+a^2+b^2,$$

which is the locus of the point of intersection of perpendicular tangents.

(4) *The locus of the pole of a given straight line with respect to a system of confocal conics is a straight line.*

Let the given line be

$$lx+my=1 \qquad \dots (1)$$

and let

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad \dots (2)$$

be any member of a given confocal system.

If (x', y') be the pole of (1) with respect to (2),

$$\frac{xx'}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}=1$$

must be identical with (1).

Hence,

$$\frac{x'}{a^2+\lambda}=l, \qquad \frac{y'}{b^2+\lambda}=m.$$

Eliminating λ from these,

$$\frac{x'}{l}-a^2=\frac{y'}{m}-b^2.$$

The locus of (x', y') is therefore the straight line

$$\frac{x}{l}-\frac{y}{m}=a^2-b^2 \qquad \dots (3)$$

which is perpendicular to (1).

Now (1) touches one member of the given confocal system. The point of contact is the pole of (1) with respect to that confocal. The locus given by (3) passes through this point.

(5) *If two conics confocal with a given conic*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

pass through a given point P, and PQ, PR the normals at P to the confocals meet the polar of P with respect to the given conic in Q and R, then

$$PQ=-\frac{\lambda_1}{p_1}, \quad PR=-\frac{\lambda_2}{p_2}$$

where p_1, p_2 are the perpendiculars from the centre to the tangents at P to the confocals, and λ_1, λ_2 the parameters of the confocals.

Let the co-ordinates of P be (α, β).

The equation to the tangent at P to the confocal

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \qquad \ldots \quad (1)$$

is

$$\frac{x\alpha}{a^2+\lambda_1}+\frac{y\beta}{b^2+\lambda_1}=1 \qquad \ldots \quad (2)$$

The equation of the normal to (1) at P is

$$\frac{\beta}{b^2+\lambda_1}(x-\alpha)-\frac{\alpha}{a^2+\lambda_1}(y-\beta)=1,$$

which can be written as

$$\frac{x-\alpha}{\dfrac{p_1\,\alpha}{a^2+\lambda_1}}=\frac{y-\beta}{\dfrac{p_1\,\beta}{b^2+\lambda_1}}=r,$$

where
$$p_1=\frac{1}{\sqrt{\dfrac{\alpha^2}{(a^2+\lambda_1)^2}+\dfrac{\beta^2}{(b^2+\lambda_1)^2}}},$$

which represents the length of the perpendicular from the centre upon the tangent to (1), and r is the distance of (x, y) from (α, β).

Hence, if $PQ=r$, the co-ordinates of Q are

$$\alpha\left(1+\frac{p_1 r}{a^2+\lambda_1}\right), \beta\left(1+\frac{p_1 r}{b^2+\lambda_1}\right).$$

But Q is on the polar of P, and therefore

$$\frac{\alpha^2}{a^2}\left(1+\frac{p_1 r}{a^2+\lambda_1}\right)+\frac{\beta^2}{b^2}\left(1+\frac{p_1 r}{b^2+\lambda_1}\right)$$
$$=1=\frac{\alpha^2}{a^2+\lambda_1}+\frac{\beta^2}{b^2+\lambda_1},$$

since P lies on (2).

This gives

$$(p_1 r+\lambda_1)\left\{\frac{\alpha^2}{a^2(a^2+\lambda_1)}+\frac{\beta^2}{b^2(b^2+\lambda_1)}\right\}=0.$$

Therefore $r=-\dfrac{\lambda_1}{p_1}$. Similarly $PR=-\dfrac{\lambda_2}{p_2}$.

14·7. Confocal Parabolas. Parabolas having a common focus and the same line for axis are said to be confocal. If the common focus is taken as the origin and the common axis the axis of x, the equation

$$y^2=4a(x+a)$$

will give a system of confocal parabolas for different values of a.

EXAMPLES ON CHAPTER XIV

1. Show how to express the co-ordinates (x_1, y_1) of a point P in terms of the parameters λ_1, λ_2 of the two confocals passing through P and belonging to the family

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

If 2φ be the angle between the two tangents from P to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

show that

$$\tan 2\varphi=\frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1+\lambda_2}.$$

2. Tangents are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from any point T on a given hyperbola confocal with the ellipse; if 2θ be the angle between the tangents, prove that $\sin\theta$ varies inversely as CD, where CD is the semi-diameter conjugate to CT of the ellipse through T confocal with the given one.

[Indian Audit and Accts. Service, 1944]

3. Two conics S_1, S_2 have a common centre (α, β); S_1 touches the x-axis at the origin, S_2 touches the y-axis

at the origin. Write down suitable equations for S_1 and S_2 and obtain the condition that they be confocal.

4. TQ, TP are tangents one to each of two confocal conics whose centre is C; if the tangents are at right angles to one another, show that CT will bisect PQ.

5. Show that the equation of the pair of tangents from P to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

referred to the normals to the confocals through P as axes

is
$$\frac{X^2}{\lambda_1}+\frac{Y^2}{\lambda_2}=0,$$

where λ_1 and λ_2 are the parameters of the confocals through P.

6. If λ, μ be the parameters of the confocals through two points P, Q on a given ellipse, show (*i*) that if P, Q be the extremities of conjugate diameters, then $\lambda+\mu$ is constant and (*ii*) that if the tangents at P and Q be at right angles, then $\frac{1}{\lambda}+\frac{1}{\mu}$ is constant.

7. Show that the locus of the points of contact of tangents drawn through a fixed point (h, k) to a series of conics confocal with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the cubic curve

$$\frac{x}{y-k}+\frac{y}{x-h}=\frac{a^2-b^2}{hy-kx},$$

which passes through the foci of the confocals.

8. Tangents are drawn to the parabola $y^2=4x\sqrt{a^2-b^2}$ and on each is taken the point at which it touches one of the confocals

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1\;;$$

prove that the locus of such points is a straight line.

9. Show that the ends of the equal conjugate diameters of a series of confocal ellipses are on a confocal rectangular hyperbola.

10. The rectangle under the perpendiculars let fall on a straight line from its pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and from the centre of the ellipse is constant $(=\lambda)$; prove that the straight line touches the confocal

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

11. Two tangents OP, OQ being drawn to a given conic, prove that two other conics can be drawn confocal with the given conic and having for their polars of O the normals at P, Q.

12. An ellipse and a hyperbola are confocal and the asymptotes of the hyperbola lie along the equiconjugate diameters of the ellipse. Prove that the hyperbola will cut orthogonally all conics which pass through the ends of the axes of the ellipse. [London, 1945]

13. An ellipse is described confocal with a given hyperbola, and the asymptotes of the hyperbola are the equal conjugate diameters of the ellipse. Prove that, if from any point of the ellipse tangents be drawn to the hyperbola, the centres of two of the circles which touch these tangents and the chord of contact will lie on the hyperbola.

14. A parabola is drawn having its axis parallel to a given straight line and having double contact with a given ellipse; prove that the locus of its focus is a hyperbola confocal with the ellipse and having one asymptote in the given direction.

15 Prove that the product of the four normals drawn to an ellipse from a point P is

$$\frac{\lambda_1\lambda_2(\lambda_1-\lambda_2)}{a^2-b^2},$$

where λ_1, λ_2 are the parameters of the confocals to the given ellipse which pass through P, and a, b the semi-axes of the given ellipse.

16. Find the locus of points whose polar lines with respect to the conics

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1 \qquad \ldots (1) \qquad (\alpha+\beta\neq 0)$$

and
$$ax^2+2hxy+by^2=1 \qquad \ldots (2)$$

are perpendicular.

Hence, or otherwise, show that the conics having centre at the origin which meet (1) orthogonally at four points are either confocal with it, or else pass through four fixed points on its principal axes.

[Math. Tripos, 1940]

ANSWERS

Pages 14—15

1. $(x_1+x_2-x_3,\ y_1+y_2-y_3),\ (x_1-x_2+x_3,\ y_1-y_2+y_3),$
$(-x_1+x_2+x_3,\ -y_1+y_2+y_3).$

Pages 47—50

1. $4x^2+4y^2=l^2.$

5. $\dfrac{h}{x}+\dfrac{k}{y}=2$, where (h, k) is the given point and the two given lines are the axes of reference.

9. (*i*) $(0, 0)$; (*ii*) $x+\sqrt{3}y=3-2\sqrt{3}$; (*iii*) $(-2\sqrt{3}, \sqrt{3})$, or $(6+2\sqrt{3}, -4-\sqrt{3})$, or $\left(4+\dfrac{2}{\sqrt{3}}, \sqrt{3}\right)$, or $\left(2-\dfrac{2}{\sqrt{3}}, -4-\sqrt{3}\right)$.

10. $4x-3y-25=0,\ 3x+4y-25=0.$

12. $y^2(a^2+p^2)=a^2(y-p)^2-p^2x^2$, where the base and its perp. bisector are taken as axes, $2a$ is the length of the base and p the altitude of the triangle.

Pages 76—80

1. k should not lie between $\sqrt{a}$ and $\sqrt{b}$.

7. $(f^2+g^2)/(a^2+h^2)$.

13. $$\frac{\{a_1(b_2c_3-c_2b_3)+b_1(c_2a_3-a_2c_3)+c_1(a_2b_3-b_2a_3)\}^2}{2(a_2b_3-b_2a_3)(a_3b_1-a_1b_3)(a_1b_2-a_2b_1)}.$$

Pages 105—108

12. $m(ax+hy+g)=l(hx+by+f).$

Pages 131—133

2. Circum-circle $(2, \sqrt{3}), 2$; in-circle

$(\sqrt{3}\tan 15^\circ+1, \sqrt{3})$; $(\sqrt{3}\tan 15^\circ+1)\dfrac{\sqrt{3}}{2}$.

3. $x+y=\pm 2\sqrt{2}$, $x-y+4\pm 2\sqrt{2}=0$.

5. 1, 5.

6. $(x^2+y^2)(a\sin^2\alpha-2h\sin\alpha\cos\alpha+b\cos^2\alpha)$
$-p[(a-b)\sin\alpha-2h\cos\alpha]x+p[(a-b)\cos\alpha$
$+2h\sin\alpha]y=0$;

$$\frac{p^2(h^2-ab)^{\frac{1}{2}}}{a\sin^2\alpha-2h\sin\alpha\cos\alpha+b\cos^2\alpha};$$

$$\frac{p^3\sqrt{(h^2-ab)\{(a-b)^2+4h^2\}}}{(a\sin^2\alpha-2h\sin\alpha\cos\alpha+b\cos^2\alpha)^2}.$$

Pages 150—153

4. (1,2), (3,1).

14. $(0,0), \left(\frac{2ab}{a+b}, 0\right)$.

Pages 188—192

1. $\sqrt{x}+\sqrt{y}=2^{\frac{3}{4}}$.

4. $y^2=a(x-3a)$.

5. $y^2=a(x-3a)$.

Pages 235—239

3. $(x^2+y^2)^2=a^2x^2+b^2y^2$.

21. $\frac{x^2}{a^2}[y^2(a^2-b^2)+b^2(a^2+b^2)]^2+\frac{y^2}{b^2}[x^2(a^2-b^2)$
$-a^2(a^2+b^2)]^2=(a^2y^2+b^2x^2)^2$.

Pages 262—266

1. $x=3, y=2$; $xy-2x-3y+12=0$.

9. $(a^2-b^2)^2\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=(a^2-b^2-x^2-y^2)^2$.

Pages 284—287

1. (*i*) Parabola ; axis $2x+2y-1=0$; tangent at the vertex $4x-4y+5=0$; latus rectum $\frac{1}{\sqrt{2}}$.

(*ii*) Parabola ; axis $4x-3y+2=0$; tangent at the vertex $3x+4y=1$; latus rectum 8.

(*iii*) Ellipse ; centre (1, 2) ; major axis $2x+y-4=0$, length 6 ; minor axis $x-2y+3=0$, length 4.

(iv) Hyperbola ; centre $(-4, 0)$; transverse axis $x+y+4=0$, length $2\sqrt{2/7}$; conjugate axis $x-y+4=0$, length $2\sqrt{2/3}$.

(v) Hyperbola ; centre $(-2, 2)$; transverse axis $x-y+4=0$, length $2\sqrt{2}$; conjugate axis $x+y=0$, length $2\sqrt{2/5}$.

(vi) Ellipse ; centre $(1, 1)$; major axis $x+2y-3=0$, length $2\sqrt{3/5}$; minor axis $2x-y-1=0$, length $2\sqrt{\frac{2}{5}}$.

(vii) Ellipse; centre $(2, -2)$; major axis $3x-2y-10=0$, length $2\sqrt{2}$; minor axis $2x+3y+2=0$, length 2.

(viii) Parabola ; axis $12x-5y+5=0$; tangent at the vertex $5x+12y-12=0$; latus rectum $\frac{4}{13}$.

(ix) Ellipse ; centre $(2, -3)$; major axis $4x+3y+1=0$, length 6 ; minor axis $3x-4y-18=0$, length 4.

(x) Parabola ; axis $3x-4y+7=0$; tangent at the vertex $4x+3y+2=0$; latus rectum 3.

(xi) Two parallel straight lines $x+2y+1=0$, $x+2y+2=0$.

2. $(-7, -4)$.

3. $\frac{1}{2\sqrt{10}}$; $(1, 1)$; $3x-y=2$; $4x+12y=15$.

4. Ellipse ; centre $(-1, 1)$; major axis $2x-y+3=0$, length 4 ; minor axis $x+2y-1=0$, length 2.

5. Transverse axis $3x-4y+2=0$, length $2\sqrt{3}$; conjugate axis $4x+3y+11=0$, length $2\sqrt{2}$; asymptotes $x^2+24xy-6y^2+28x+36y+46=0$.

6. Hyperbola ; centre $(1, 0)$; transverse axis $x-2y-1=0$, length 4 ; conjugate axis $2x+y-2=0$, length 6 ; foci $(1+2\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{5}})$, $(1-2\sqrt{\frac{13}{5}}, -\sqrt{\frac{13}{5}})$.

7. Hyperbola; centre $(3, 4)$; transverse axis $4x+3y=24$, length 2 ; conjugate axis $3x-4y+7=0$, length $4\sqrt{6}$; eccentricity 5.

8. Ellipse ; centre $(1, -2)$; major axis $3x-2y-7=0$, length 6 ; minor axis $2x+3y+4=0$, length 4.

12. Hyperbola ; centre $(-1, 2)$; transverse axis $x+2y=3$, length $\sqrt{2}$; conjugate axis $2x-y+4=0$, length $\frac{2}{\sqrt{3}}$; asymptotes $x^2-4xy-2y^2+10x+4y+1=0$.

13. Ellipse ; centre $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; major axis $x+y=\sqrt{2}$, length $2\sqrt{2}$; minor axis $x-y=0$, length 2 ; foci $(0, \sqrt{2})$, $(\sqrt{2}, 0)$.

14. $x-2y+1=0$; $2x+y=3$.

15. $\left(\frac{a'b-ab'-k(b+b')}{b-b'}, 0\right)$; $y(a+k-a')+x(b'-b)$
$+a'b-ab'-k(b+b')=0$.

16. Hyperbola ; centre $(2, 3)$; transverse axis $4x-3y+1=0$, length 6 ; conjugate axis $3x+4y=18$, length 4.

Pages 301—304

7. $\frac{l}{r}=\sec\alpha+e\cos\theta$.

9. $\frac{l}{r}=\cos\alpha+e\cos\theta$; the conic is an ellipse, parabola or hyperbola according as $\cos\alpha \gtreqless e$. **15.** $2l\sec\beta$.

Pages 339—342

3. $S_1\equiv a'\beta x^2-2a'\alpha xy+b'\beta y^2-2(b'\beta^2-a'\alpha^2)y=0$,

$S_2\equiv a\alpha x^2-2b\beta xy+b\alpha y^2-2(a\alpha^2-b\beta^2)x=0$;

$$\left(\frac{a-b}{a'-b'}\right)^2=\frac{b^2\beta^4}{a'^2\alpha^4}.$$

16. $a\beta x^2+(\alpha+\beta)hxy+b\alpha y^2=0$.